National Seminar on Control and Optimization
Nov 13-14, 2019
سومین سمینار ملی
کنترل و بهینه‌سازی
۲۲-۲۳ آبان ۱۳۹۸

DECOMPOSITION OF THE MULTIOBJECTIVE OPTIMIZATION PROBLEM AND THE CONCEPT OF EQUITABLE EFFICIENCY

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Abstract. In this paper, we decomposed the vector of objective functions of the multiobjective optimization problem into a collection of smaller-sized subproblems. In describing the relationships between solutions of the original and decomposed problems, it is shown that the set of equitably efficient solutions of the subproblems is contained within the set of efficient solutions for the original problem. Furthermore, by using the concept of P-equitable efficiency two new multiobjective optimization problems are presented to coordinate equitably efficient solutions of subproblems.

1. Introduction

The equitable preference was first known as the generalized Lorenz dominance [2, 4]. Kostreva and Ogryczak [1] are the first ones who introduced the concept of equitability into multiobjective programming. In equitable multiobjective optimization all the objectives are uniformly optimized, but in some cases the decision maker believes that some of them should be uniformly optimized according to the importance of objectives. To solve this problem in this paper, the original problem is decomposed into a collection of smaller subproblems, according to the decision maker, and the subproblems are solved by the concept of P-equitable efficiency. For more details, the reader may refer to [3].

Throughout this article the following notation is used. Let R^m be the Euclidean vector space and $y', y'' \in R^m$. $y' \leqq y''$ denotes $y'_i \leq y''_i$ for all $i = 1, 2, \cdots, m$. $y' < y''$ denotes $y'_i < y''_i$ for all $i = 1, 2, \cdots, m$. $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$.

2010 *Mathematics Subject Classification.* 90C29, 91B08, 90B50.
Key words and phrases. Pareto, Equitability, Decomposition, Coordination, Multiobjective programming.

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Consider a decision problem defined as an optimization problem with m objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{aligned}&\min\left(f_1(x), f_2(x), \cdots, f_m(x)\right),\\&\textit{subject to } x \in X,\end{aligned} \tag{1.1}$$

where x denotes a vector of decision variables selected from the feasible set X and $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space R^m. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.

2. Main results

In this section, we will introduce the concept of P-equitably efficient solution by decomposition of objective functions.

Let $P \subset \{1, 2, \cdots, m\}$ and denote by $f_P = (f_j)_{j\in P}$ the objective functions vector that only contains f_j, $j \in P$. Suppose that n be a positive integer such that $n \leq m$. If $P = \{P_1, P_2, \ldots, P_n\}$ is a partition of $\{1, 2, \cdots, m\}$, then the multiobjective problem

$$\begin{aligned}&\min f_{P_k}(x) \qquad\qquad (k = 1, 2, \cdots, n)\\&\textit{subject to } x \in X\end{aligned} \tag{2.1}$$

is called a subproblem of the multiobjective problem (1.1) and the collection of all these subproblems is called a decomposition of the multiobjective problem (1.1).

Definition 2.1. Let $y \in R^m$.

1. Let $\Theta : R^m \to R^m$ be the ordering map defined as $\Theta(y) = (\theta_1(y), \theta_2(y), \cdots, \theta_m(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \cdots \geq \theta_m(y)$, $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \cdots, m$, and τ is a permutation of the set $\{1, 2, \cdots, m\}$.
2. Let $\overline{\Theta} : R^m \to R^m$ be the cumulative ordering map defined as
$$\overline{\Theta}(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \cdots, \bar{\theta}_m(y)),$$
where $\bar{\theta}_i(y) = \sum_{j=1}^{i} \theta_j(y)$ for $i = 1, 2, \cdots, m$.

Definition 2.2. Let $y', y'' \in Y$ be two outcome vector. We say that y', P-equitably dominates y'' and denoted by $y' \prec_P y''$, iff

$$\overline{\Theta}(y'_{P_k}) \leqq \overline{\Theta}(y''_{P_k}) \qquad (k = 1, 2, \cdots, n),$$

and $\overline{\Theta}(y'_{P_k}) \leq \overline{\Theta}(y''_{P_k})$ for some $k \in \{1, 2, \cdots, n\}$.

Definition 2.3. We say that outcome vector $y \in Y$ is P-equitably nondominated iff there does not exist $y' \in Y$ such that $y' \prec_P y$. Also we say that feasible solution $x \in X$ is a P-equitably efficient solution of the multiobjective problem (1.1) iff $y = f(x)$ is P-equitably nondominated.

Note that when $n = 1$ namely $P_1 = \{1, 2, \cdots, m\}$, the relation $\prec_P$ becomes the equitable efficiency relation, $\prec_e$. The above definition permits one to express P-equitable efficiency for problem (1.1) in terms of the standard efficiency for the multiobjective problem

$$min \;\; \left\{ \left(\overline{\Theta}(f_{P_1}(x)), \overline{\Theta}(f_{P_2}(x)), \cdots, \overline{\Theta}(f_{P_n}(x)) \right) : x \in X \right\}. \tag{2.2}$$

Theorem 2.4. *The feasible solution $x \in X$ is a efficient solution of the multiobjective problem (2.2) if and only if it is a P-equitably efficient solution of the multiobjective problem (1.1).*

Remark 2.5. If $n = 1$ namely $P_1 = \{1, 2, \cdots, m\}$, we have Corollary 2.2 from [1]. So, the feasible solution $x \in X$ is an efficient solution of the multiobjective problem

$$min \;\; \left\{ \overline{\Theta}(f(x)) \; : \; x \in X \right\}, \tag{2.3}$$

if and only if it is an equitably efficient solution of the multiobjective problem (1.1).

Note that if x is a P-equitably efficient solution of multiobjective problem (1.1), then it is also Pareto-optimal solution for this problem. Therefore, to reduce Pareto-optimal solution, we can use P-equitably efficient solution.

Theorem 2.6. *Suppose that $k \in \{1, 2, \cdots, n\}$. If $\sum_{j \in P_k} f_j$ is an injective function and if $x \in X$ is an efficient solution of the multiobjective problem*

$$min\{\overline{\Theta}(f_{P_k}(x)) : x \in X\}, \tag{2.4}$$

then it is an efficient solution of the multiobjective problem (2.2).

According to Remark 2.5, each equitably efficient solution of the multiobjective problem (2.1) is also an efficient solution of the multiobjective problem (2.4). By using Theorem 2.6 and Theorem 2.4, the following result is obtained.

Corollary 2.7. *Suppose that $k \in \{1, 2, \cdots, n\}$. If $\sum_{j \in P_k} f_j$ is an injective function and if $x \in X$ is an equitably efficient solution of the multiobjective problem (2.1), then it is a P-equitably efficient solution of the multiobjective problem (1.1).*

Since the set of efficient solutions of the multiobjective problem (2.2) is contained within the set of efficient solution of the multiobjective problem (1.1), and the set of efficient solutions of the multiobjective problem (2.2) contains the set of equitably efficient solutions of the multiobjective problem (2.1), is obtained from solving the multiobjective problem (2.4) for $k = 1, 2, \cdots, n$, we can use efficient solutions of the multiobjective problem (2.2) to coordinate equitably efficient solutions of subproblems.

Scalarization is one of the most common approaches used to solve a multiobjective problem. Scalarizing functions are used to transform a given multiobjective problem into a single objective optimization problem, by aggregating the objectives of a multiobjective problem into a single objective. Typical solution concepts for multiobjective problems are defined by scalarizing functions $s : Y \to \mathbb{R}$ to be minimized. Thus the multiobjective problem (1.1) is replaced with the minimization problem

$$\min\{s(f(x)) : \quad x \in X\}. \tag{2.5}$$

The preference relation corresponding to the problem (2.5) is defined as follows:

$$y' \preceq y'' \Leftrightarrow s(y') \leq s(y'').$$

For any strictly convex, increasing function $g : R \to R$, the scalarizing function is defined by

$$s(y) = \sum_{i=1}^{m} g(y_i)$$

is a strictly monotonic and strictly Schur-convex function [4]. It has been shown in Proposition 3.1 from [1], the preference relation corresponding to this scalarizing function is an equitable rational preference relation. Also, every optimal solution of the problem (2.5) is an equitably efficient solution of the original multiobjective problem. Similarly, if $g_k : R \to R$ for $k = 1, 2, \cdots, n$ be strictly convex, increasing function, then the optimal solution of the problem

$$\min \left\{ \sum_{i \in P_k} g_k(f_i(x)) : x \in X \right\}, \tag{2.6}$$

is an equitably efficient solution of the multiobjective problem (2.1).

Theorem 2.8. *Suppose that $k \in \{1, 2, \cdots, n\}$ and $x \in X$ is a feasible solution. If $\sum_{i \in P_k} (g_k \circ f_i)$ is an injective function and if x is an optimal solution of the problem (2.6) then it is an efficient solution of the multiobjective problem*

$$\min \left\{ \left(\sum_{i \in P_1} g_1(f_i(x)), \sum_{i \in P_2} g_2(f_i(x)), \cdots, \sum_{i \in P_n} g_n(f_i(x)) \right) : \quad x \in X \right\}. \tag{2.7}$$

Finally, we will generate the P-equitably efficient solutions by introducing certain scalarizing functions.

Theorem 2.9. *Let $x \in X$ be a feasible solution and let $g_k : R \to R$ be a strictly convex, increasing function for $k = 1, 2, \cdots, n$. If x is an efficient solution of the multiobjective problem (2.7), then it is a P-equitably efficient solution of the multiobjective problem (1.1).*

Remark 2.10. If $n = 1$ namely $P_1 = \{1, 2, \cdots, m\}$, we have Corollary 3.1 from [1].

Since the set of efficient solutions of the multiobjective problem (2.7) is contained within the set of efficient solutions the multiobjective problem (1.1), and the set of efficient solutions of the multiobjective problem (2.7) contains the set of equitably efficient solutions of the multiobjective problem (2.1), is obtained from solving problem (2.6) for $k = 1, 2, \cdots, n$, we can use efficient solutions of the multiobjective problem (2.7) to coordinate equitably efficient solutions of subproblems.

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