National Seminar on Control and Optimization
3th
سومین سمینارملی
کنترل وبهینه سازی

A PROJECTION RNN FOR NON-CONVEX OPTIMIZATION PROBLEMS

MOHAMMAD ESHAGHNEZHAD[1]*, SOHRAB EFFATI[1], AND AMIN MANSOORI[1]

[1] *Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.*
m_shaghnezhad@yahoo.com; s-effati@um.ac.ir; am.ma7676@yahoo.com

Abstract. The present scientific attempt is devoted to investigating the nonconvex optimization problems (NCOPs) utilizing the concepts of projection Recurrent Neural Networks (RNN)s. For this purpose, the original problem is reformulated into a m-th power form. Then, the Karush–Kuhn–Tucker (KKT) optimality conditions are provided. The KKT conditions are used to propose the RNN model. Besides, the Lyapunov stability and the global convergence of the RNN model are proved.

1. Introduction

Many problems in engineering, such as optimal control, adaptive signal processing, kinematic control of redundant robot manipulators, and non-linear model predictive control can be modelled as dynamic programming problems and many of them are involved with non-convexity and multiplicity in objective functions. For example, control of redundant robot manipulators and real-time motion planning can be modelled as constrained NCOPs for simultaneously maximizing manipulability and minimizing kinetic energy. Obtaining the real time optimal solutions is the difficulty of dynamic optimization, especially in the presence of uncertainty. Applying the RNN models for optimization in such applications are always more competent than conventional optimization techniques. This is because of their salient features of biological plausibility, intrinsic nature of distributed and parallel information processing and hardware parallelizability.

The pioneering works on an RNN model to optimization is for Hopfield and Tank [1]. Neurodynamic optimization has received great success in recent years [2, 3, 5, 4]

Key words and phrases. Non-convex optimization problems, recurrent neural network, Lyapunov stability.

* Speaker.

2. Non-convex optimization

In this section we introduce the non-convex optimization problem.
Consider the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(x), \\ s.t. \quad & g_j(x) \le b_j, \quad j = 1, 2, \ldots, m, \\ & x \in X, \end{aligned} \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ are m-dimensional vector-valued functions of n variables, and X is a box set defined as $X = \{x \in \mathbb{R} \mid u \le x \le v, \ u, v \in \mathbb{R}\}$. In this paper, the functions $f, g_1(x), \ldots, g_m(x)$ are assumed to be twice differentiable. Also, we assume that f is positive on X, g_j is non-negative on X, and b_j is positive for $j = 1, \ldots, m$. Note that, the following notations. $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$. $\mathbb{R}^n_+$ stands for the non-negative quadrant in the n-dimensional real space. int$X$ stands for the interior of a set X. The Lagrangian function for problem (2.1) and the Hessian of the Lagrangian can be formulated as follows:

$$\begin{aligned} L(x, \lambda) &= f(x) + \sum_{j=1}^{m} \lambda_j (g_j(x) - b_j), \ \lambda = (\lambda_1, \ldots, \lambda_m) \ge 0 \\ \nabla_x^2 L(x, \lambda) &= \nabla^2 f(x) + \sum_{j=1}^{m} \lambda_j \nabla^2 g_j(x). \end{aligned} \tag{2.2}$$

Lemma 2.1 (Second-order sufficiency conditions [6])**.** *Suppose that x^* is a feasible and regular point of the problem (2.1). If there exists $\lambda^* \in \mathbb{R}^m$, such that (x^*, λ^*) is a KKT point pair and the Hessian matrix, that is (2.2) is positive definite on the tangent subspace $M(x^*) = \{d \in \mathbb{R}^n \mid d^T \nabla g_j(x^*) = 0, \ d \ne 0, \ \forall j \in J(x^*)\}$, where $J(x^*)$ is defined by $J(x^*) = \{j \in J \mid \lambda_j > 0\}$, then x^* is a strict minimum of problem (2.1).*

If in problem (2.1) all functions $f(x)$ and $g_j(x)$ are convex over the box set X, the problem is called a convex optimization problem; otherwise, it is called a non-convex optimization problem. In this point of view, let problem (2.1) be a non-convex optimization problem. Consider the partial p-power transformation of problem (2.1) as follows:

$$\begin{aligned} \min \quad & [f(x)]^p, \\ s.t. \quad & [g_j(x)]^p \le b_j^p, \quad j = 1, 2, \ldots, m, \\ & x \in X. \end{aligned} \tag{2.3}$$

The Lagrangian function of problem (2.3) is defined as,

$$L_p(x, \mu) = [f(x)]^p + \sum_{j=1}^{m} \mu_j ([g_j(x)]^p - b_j^p),$$

where $\mu = (\mu_1, \ldots, \mu_m) \ge 0$. Also, the Lagrange multipliers are defined in [7]. Here, we state some results about the non-convex optimization problem and it's partial p-power transformation.

Lemma 2.2 ([7])**.** *Let x^* be a local optimal solution of (2.1) in* $\text{int}X$*. Assume that, x^* is a regular point and satisfies the second-order sufficiency conditions. If $J(x^*) \neq \emptyset$, then there exists a $q > 0$ such that the Hessian of the partial p-power Lagrangian function, that is $\nabla_x^2 L_p(x^*, \mu^*)$, is positive definite when $p > q$.*

Theorem 2.3 ([7])**.** *Let $x^* \in \text{int}X$ be a local optimal solution of (2.1). If all conditions in Lemma 2.2 hold, then there exists a $q > 0$ such that $\nabla_x^2 L_p(x^*, \mu^*)$ is positive definite on a set $N \subset X \times \mathbb{R}_+^m$ when $p > q$. Note that, N is a set on which $\nabla_x^2 L(x, \lambda)$ or $\nabla^2 f(x)$ is positive definite.*

According to the former discussion, since $\nabla_x^2 L_p(x^*, \mu^*)$ is positive definite.

3. Projection RNN model

In this section, we propose our new neurodynamic model.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous vector function with constant $L > 0$. Consider the function $g(x, \beta)$ as follow for $0 < \beta < \frac{1}{L}$:

$$g(x, \beta) = e(x, \beta) - \beta[F(x) - F(x - e(x, \beta))], \tag{3.1}$$

$$e(x, \beta) = x - P_\Omega[x - \beta F(x)]. \tag{3.2}$$

We propose the new neurodynamic model as follows:

$$\frac{dx}{dt} = P_\Omega[x - \lambda g(x, \beta)] - x, \tag{3.3}$$

where $0 < \lambda \leq 1$.

Theorem 3.1. *Assume that, $F(\cdot)$ is a Lipschitz continuous function in $\mathbb{R}^n$. Then, there is a unique Lipschitz continuous solution $x(t)$ for (3.3). Moreover, when $x(t_0) \notin \Omega$, the solution $x(t)$ approaches Ω, exponentially. Besides, when $x(t_0) \in \Omega$, $x(t) \in \Omega$.*

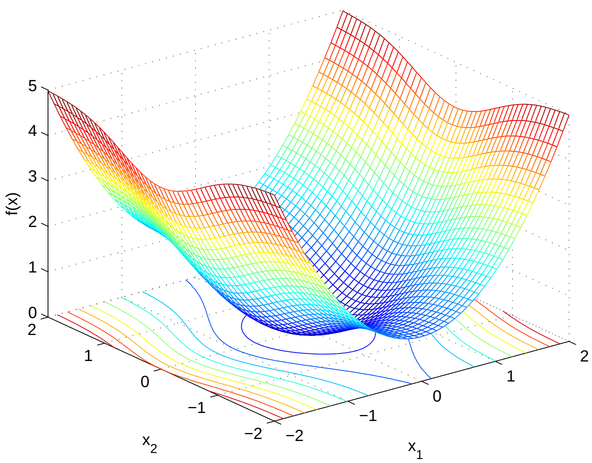
Theorem 3.2. *Let $F(\cdot)$ be a pseudo-monotone and Lipschitz continuous function with the constant L. Then, the neurodynamic model (3.3) for $0 < \beta \leq \frac{1}{5L}$ and $0 < \lambda \leq 1$ is stable in the sense of Lyapunov and globally convergent.*

4. An example

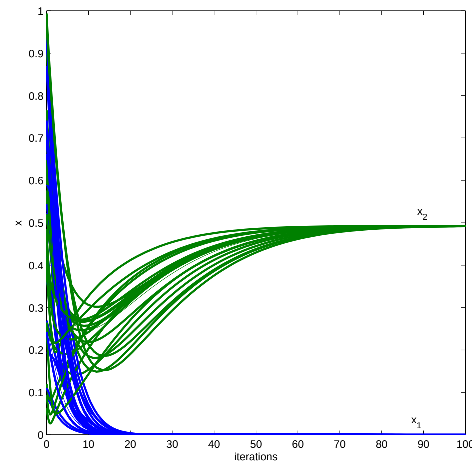
Example 4.1 ([8])**.** Consider an invex optimization problem as follows:

$$\begin{aligned} \min \quad & f(x) = 1 + x_1^2 - \exp(-x_2^2) \\ s.t. \quad & g_1(x) = x_1^2 - x_2 + 0.5 \leq 0 \\ & g_2(x) = 2x_2 - x_1^2 - 3 \leq 0. \end{aligned}$$

Figure 1 shows the isometric view of $f(x)$ in 3-D space. As depicted in Figure 1(A), the objective function $f(x)$ is a smooth invex function. The feasible region $S = \{x \in \mathbb{R}^2,\ x_1^2 - x_2 + 0.5 \leq 0,\ 2x_2 - x_1^2 - 3 \leq 0\}$ is not a convex set. The invex optimization problem has a unique KKT point $(0, 0.5)$, which is the global minimum solution. Therefore, we use our model to solve this problem by letting $\lambda = 0.5$ and $\beta = 0.05$. Figure 1(B) shows the convergence behaviour of the neurodynamic model (3.3).



(A) Isometric view of the objective function in Example 4.1.



(B) Trajectories behaviour of neural network (3.3) in Example 4.1.

Figure 1. Example 4.1.

References

1. J. J. Hopfield, D. W. Tank, Neural computation of decisions in optimization problems, Biological Cybernetics, 52 (1985) 141–152.
2. M. Eshaghnezhad, S. Effati, A. Mansoori, A Neurodynamic Model to Solve Nonlinear Pseudo-Monotone Projection Equation and Its Applications, IEEE Transactions on Cybernetics, 47 (10) (2017) 3050–3062.
3. A. Mansoori, M. Eshaghnezhad, S. Effati, An efficient neural network model for solving the absolute value equations, IEEE Transactions on Circuits and Systems II: Express Briefs, 65 (3) (2018) 391–395.
4. M. Mohammadi, A. Mansoori, A Projection Neural Network for Identifying Copy Number Variants, IEEE journal of biomedical and health informatics, 23 (5) (2018) 2182–2188.
5. A. Mansoori, M. Eshaghnezhad, S. Effati, Recurrent Neural Network Model: A New Strategy to Solve Fuzzy Matrix Games, IEEE Transactions on Neural Networks and Learning Systems, 30 (8) (2019) 2538 - 2547.
6. D. G. Luenberger, Linear and Nonlinear Programming, 2nd Edition. Addison- Wesley, Reading, Massachusetts, 1984.
7. D. Li and X. L. Sun, Local convexification of the Lagrangian function in nonconvex optimization, Journal of Optimization Theory and Applications, 104 (1) (2000) 109–120.
8. G. Li, Z. Yan, and J. Wang, A one-layer recurrent neural network for constrained nonconvex optimization, Neural Networks, 61 (2015) 10–21.