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ON THE STABILIZATION OF A COUPLED SYSTEM OF FRACTIONAL ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the stabilization problem of a cascade of a fractional ordinary differential equation (FODE) and a fractional diffusion (FD) equation where the interconnections are of Neumann type. We exploit the PDE backstepping method as a powerful tool for designing a controller to show the Mittag-Leffler stability of the FD-FODE cascade. Finally, a numerical example is presented to verify the results.

1. Introduction

In control engineering, control of partial differential equations (PDEs) is physically motivated and is a challenging subject for research [2]. The stabilizability of the systems described by PDEs is hard to check [6], so an efficient tool is required to analyze the stability of the PDE systems.

One of the most useful approaches for boundary controller design is the PDE backstepping method [2]. On the other hand, real systems in our world are complex and can be well characterized by fractional order's notions [5]. In [1] the backstepping-based boundary feedback control problem of fractional reaction diffusion system with mixed or Robin boundary control is addressed.

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In the last decades, the stabilization problem for coupled systems becomes one of the challenging issues. The cascade structure for the heat PDE with an ODE, when the interconnection is of Dirichlet type, is discussed in [3]. In [4], the stabilization problem for a new cascade of PDE-ODE is studied.

To the best of our knowledge, the stabilization problem and also designing the controller for a cascade of an FD equation and an FODE equation has not been addressed yet. In this paper, we consider a cascade of an FODE equation and an FD equation, and we use an invertible integral transformation to transfer the original system to a Mittag-Leffler stable target system. Finally, we present a numerical example to verify the theoretical results.

2. Preliminaries

Definition 2.1. The Caputo fractional-order derivative is defined by

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{x^n(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,\ \ (n-1<\alpha<n) \tag{2.1}$$

where α is the fractional order, and the gamma function Γ is defined as $\gamma(\tau)=\int_0^\infty t^{\tau-1}e^{-t}dt$.

Definition 2.2. (Mittag-Leffler stability) The solution of

$$ {}_{t_0}^{C}D_t^{\alpha}u(t) = f(t,u),$$

is said to be Mittag-Leffler stable if

$$\|u(t)\| \leq \left(m[u(t_0)]E_\alpha(-\lambda(t-t_0)^\alpha)\right)^b,$$

where t_0 is the initial value of time, $\alpha\in(0,1)$, $\lambda\geq 0$, $b>0$, $m(0)=0$, $m(u)$ is nonnegative and meets locally Lipschitz condition on $u\in\mathbb{B}\subset\mathbb{R}^n$.

3. Problem statement and analysis

Consider the cascade of a fractional diffusion (FD) equation and a fractional-order ordinary differential equation (FODE) with Caputo derivative as follows:

$$ {}_{0}^{C}D_t^{\alpha}X(t) = AX(t)+Bu_x(0,t), \tag{3.1}$$

$$ {}_{0}^{C}D_t^{\alpha}u(x,t) = u_{xx}(x,t), \tag{3.2}$$

$$ u(0,t)=0, \tag{3.3}$$

$$ u(D,t)=U(t), \tag{3.4}$$

where $0<\alpha<1$, $X(t)\in\mathbb{R}^n$ is the state of the FODE system, governed by the FODE equation, and $u(x,t)$ is the state of the FD system, governed by the FD equation, and $U(t)$ is a scalar input. Note that $t\geq 0$, $x\in[0,D]$ in which $D>0$ is the length of the PDE domain. The aim is to Mittag-Leffler stabilize the system (3.1)-(3.4).

We use the PDE backstepping approach which applies the following invertible integral transformation:

$$ w(x,t)=u(x,t)-\int_0^x q(x,y)u(y,t)dy-\gamma(x)X(t), \tag{3.5}$$

to convert the cascade of an FD and an FODE (3.1)-(3.4) into the target system given by:

$$ {}_{0}^{C}D_{t}^{\alpha}X(t) = (A+BK)X(t) + Bw_{x}(0,t), \tag{3.6}$$

$$ {}_{0}^{C}D_{t}^{\alpha}w(x,t) = w_{xx}(x,t), \tag{3.7}$$

$$ w(0,t) = 0, \tag{3.8}$$

$$ w(D,t) = 0. \tag{3.9}$$

The control gain K is chosen such that the Mittag-Leffler stability of the target system is guaranteed. After some calculations, for converting (3.1)-(3.4) to (3.6)-(3.9), it is concluded that:

$$ \gamma''(x) = A\gamma(x),\ \gamma(0) = 0,\ \gamma'(0) = K, \tag{3.10}$$

and

$$ q_{xx}(x,y) = q_{yy}(x,y),\ q(x,x) = 0,\ q(x,0) = \gamma(x)B. \tag{3.11}$$

Moreover, we consider the following assumption in the throughout of the paper:
(H1) We assume that the system (3.1) is controllable.

Theorem 3.1. *Consider a closed-loop system consisting of the plant (3.1)-(3.4) and the control law:*

$$ U(t) = K\begin{bmatrix}0_n & I_n\end{bmatrix}\left\{ e^{\begin{bmatrix}0_n & A\\ I_n & 0_n\end{bmatrix}D}\begin{bmatrix}I_n\\ 0_n\end{bmatrix}X(t) + \int_0^D e^{\begin{bmatrix}0_n & A\\ I_n & 0_n\end{bmatrix}(D-y)}\begin{bmatrix}I_n\\ 0_n\end{bmatrix}Bu(y,t)dy\right\}. \tag{3.12}$$

Assume that there exist positive constants d and β and also a symmetric positive definite matrix P, such that the control gain K satisfies in the following inequality:

$$ \Omega = \begin{bmatrix} P(A+BK)+(A+BK)^TP & PB & 0 & 0\\ B^TP & -d & 0 & 0\\ 0 & 0 & -\beta & 0\\ 0 & 0 & 0 & -\beta \end{bmatrix} \prec 0. \tag{3.13}$$

Also, suppose that $u_x(.,t)$ is square integrable and $w_x(.,t)$ is continuously differentiable for all $t \in [0,\infty]$. Then the closed-loop system under the control law (3.12) is Mittag-Leffler stable in the sense of the following norm:
$(|X(t)|^2 + \int_0^D u_x^2(x,t)dx)^{\frac{1}{2}}$

4. Numerical simulation

In this section, we present a numerical example to verify our theoretical results.

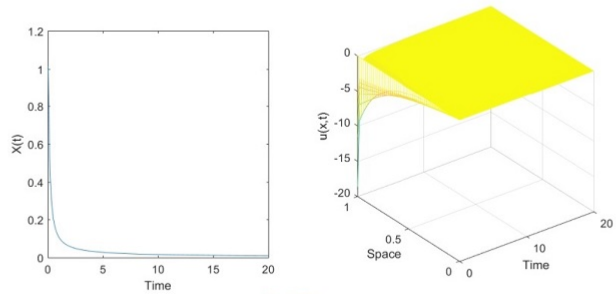

FIG. 1. Evolution of states in example 6.1. for $K = -16.0150$.

Example 4.1. Consider the following system:

$$ {}_0^C D_t^\alpha X(t) = X(t) + u_x(0,t) \tag{4.1} $$

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) \tag{4.2} $$

$$ u(0,t) = 0 \tag{4.3} $$

$$ u(D,t) = U(t) \tag{4.4} $$

in which $\alpha = 0.75$, $u(x,0) = 0$ and $X(0) = 1$ as initial conditions. $U(t)$ is determined by relation (3.12) as follows:

$$ U(t) = K\left[sinh(D)X(t) + \int_0^D sinh(D-y)u(y)dy\right] \tag{4.5} $$

we have used CVX 1.2.1 and obtain the feedback gain $K = -16.0150$ to satisfy in (3.13).

5. Conclusions

In this article we provide a Dirichlet backstepping-based boundary feedback control to guarantee the Mittag-Leffler stability of the FD-FODE coupled system. Also, the numerical example confirms the obtained results.

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