National Seminar on Control and Optimization
Nov 13-14, 2019
سومین سمینار ملی
کنترل و بهینه‌سازی
۲۲-۲۳ آبان ۱۳۹۸

ON THE GENERALIZED F-SUZUKI-CONTRACTIONS

HAMID MEHRAVARAN[1]*, REZA ALLAHYARI[1]

[1] *Department of Mathematics, Mashhad Branch, Islamic Azad University, ,
Mashhad, Iran.*
hamid_mehravaran@mshdiau.ac.ir; rezaallahyari@mshdiau.ac.ir

ABSTRACT. In this paper, we introduce the new notion of generalized F-Suzuki-contraction in the setup of dislocated S_b-metric spaces. We establish some fixed point theorems involving this contraction in complete dislocated S_b-metric spaces. We also furnish an example to verify the effectiveness and applicability of our results.

1. Introduction

In 2012, Sedghi et al. [5] introduced the concept of S-metric space by modifying D-metric and G-metric spaces and proved some fixed point theorems for a self-mapping on a complete S-metric space. After that Özgür and TaŞ studied some generalizations of the Banach contraction principle on S-metric spaces. Sedghi et al. [4] introduced the concept of S_b-metric space as a generalization of S-metric space and proved some coupled common fixed point theorems in S_b-metric space .

On the other hand, Wardowski [6] introduced a new contraction, the so-called F-contraction, and obtained a fixed point result as a generalization of the Banach contraction principle. Thereafter, Dung and Hang studied the notion of a generalized F-contraction and established certain fixed point theorems for such mappings. Recently, Piri and Kumam [2] extended the fixed point results of [6] by introducing a generalized F-Suzuki-contraction in b-metric spaces.

Motivated by the aforementioned works, in this paper, we introduce the new notion of generalized F-Suzuki-contraction in the setup of dislocated S_b-metric spaces. We establish some fixed point theorems involving this contraction in complete dislocated S_b-metric spaces. We also furnish an example to verify the effectiveness and applicability of our results.

2010 *Mathematics Subject Classification.* Primary 47H09, Secondary 47H10.

Key words and phrases. Dislocated metric space, Fixed point, Generalized F-Suzuki-contraction, S_b-Metric space.

* Speaker.

We begin with some basic well-known definitions and results which will be used further on. Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively.

Definition 1.1. [5] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to \mathbb{R}_+$ that satisfies the following conditions:

($S1$) $0 < S(x, y, z)$ for each $x, y, z \in X$ with $x \neq y \neq z \neq x$,
($S2$) $S(x, y, z) = 0$ if and only if $x = y = z$,
($S3$) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for each $x, y, z, a \in X$.

Then the pair (X, S) is called an S-metric space.

Definition 1.2. [4] Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to \mathbb{R}_+$ satisfies:

(S_b1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,
(S_b2) $S_b(x, y, z) = 0$ if and only if $x = y = z$,
(S_b3) $S_b(x, y, z) \leq b\big(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)\big)$ for all $x, y, z, a \in X$.

Then S_b is called an S_b-metric on X and the pair (X, S_b) is called an S_b-metric space.

Definition 1.3. [4] If (X, S_b) is an S_b-metric space, a sequence $\{x_n\}$ in X is said to be:

(1) Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$.
(2) convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \varepsilon$ or $S_b(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, and we denote by $\lim\limits_{n\to\infty} x_n = x$.

Definition 1.4. [4] An S_b-metric space (X, S_b) is called complete if every Cauchy sequence is convergent in X.

Example 1.5. [3] Let $X = \mathbb{R}$. Define $S_b : X^3 \to \mathbb{R}_+$ by $S_b(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, S_b) is a complete S_b-metric space with $b = 2$.

Definition 1.6. [6] Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to \mathbb{R}$ such that:

($F1$) F is strictly increasing, that is for all $\alpha, \beta \in (0, +\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,
($F2$) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim\limits_{n\to+\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to+\infty} F(\alpha_n) = -\infty$,
($F3$) there exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In 2014, Piri and Kumam described a large class of functions by replacing the condition ($F3$) in the above definition with the following one:

($F3'$) F is continuous on $(0, +\infty)$.

They denote by $\mathfrak{F}$ the family of all functions $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions ($F1$), ($F2$), and ($F3'$).
We use $\mathfrak{F}_G$ to denote the set of all functions $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions ($F1$) and ($F3'$) and Ψ to denote the set of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is continuous and $\psi(t) = 0$ if and only if $t = 0$ (see [2]).

Definition 1.7. [2] Let (X,d) be a b-metric space. A self-mapping $T:X\to X$ is said to be a generalized F-Suzuki-contraction if there exists $F\in\mathfrak{F}_G$ such that, for all $x,y\in X$ with $x\neq y$,

$$\frac{1}{2s}d(x,Tx)<d(x,y)\Rightarrow F\big(s^5d(Tx,Ty)\big)\leq F\big(M_T(x,y)\big)-\psi\big(M_T(x,y)\big),$$

in which $\psi\in\Psi$ and

$$M_T(x,y)=\max\Big\{d(x,y),d(T^2x,y),\frac{d(Tx,y)+d(x,Ty)}{2s},\frac{d(T^2x,x)+d(T^2x,Ty)}{2s},$$
$$d(T^2x,Ty)+d(T^2x,Tx),d(T^2x,Ty)+d(Tx,x),d(Tx,y)+d(y,Ty)\Big\}.$$

Theorem 1.8. [2] *Let (X,d) be a complete b-metric space and $T:X\to X$ be a generalized F-Suzuki-contraction.Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}$ converges to x^*.*

2. Main results

In this section, we first introduce the concept of dislocated S_b-metric space and then we demonstrate a fixed point result for generalized F-Suzuki-contractions in such spaces.

Definition 2.1. [1] Let X be a nonempty set and $b\geq 1$ be a given real number. A mapping $S_b:X^3\to\mathbb{R}_+$ is a dislocated S_b-metric if, for all $x,y,z,a\in X$, the following conditions are satisfied:

(dS_b1) $S_b(x,y,z)=0$ implies $x=y=z$,
(dS_b2) $S_b(x,y,z)\leq b\big(S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a)\big)$.

A dislocated S_b-metric space is a pair (X,S_b) such that X is a nonempty set and S_b is a dislocated S_b-metric on X. In the case that $b=1$, S_b is denoted by S and it is called dislocated S-metric, and the pair (X,S) is called dislocated S-metric space.

Definition 2.2. [1] Let (X,S_b) be a dislocated S_b-metric space, $\{x_n\}$ be any sequence in X and $x\in X$. Then:

- (i) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X,S_b) if, for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $S_b(x_n,x_n,x_m)<\varepsilon$ for each $m,n\geq n_0$.
- (ii) The sequence $\{x_n\}$ is said to be convergent to x if, for each $\varepsilon>0$, there exists a positive integer n_0 such that $S_b(x,x,x_n)<\varepsilon$ for all $n\geq n_0$ and we denote it by $\lim\limits_{n\to\infty}x_n=x$.
- (iii) (X,S_b) is said to be complete if every Cauchy sequence is convergent.

The following example shows that a dislocated S_b-metric need not be a dislocated S-metric.

Example 2.3. [1] Let $X=\mathbb{R}_+$, then the mapping $S_b:X^3\to\mathbb{R}_+$ defined by $S_b(x,y,z)=x+y+4z$ is a complete dislocated S_b-metric on X with $b=2$. However, it is not a dislocated S-metric space. Indeed, we have

$$4=S_b(0,0,1)\not\leq 2S_b(0,0,0)+S_b(1,1,0)=2.$$

Definition 2.4. [1] Let (X, S_b) be a dislocated S_b-metric space. A mapping $T : X \to X$ is said to be a generalized F-Suzuki-contraction if there exists $F \in \mathfrak{F}$ such that for all $x, y \in X$

$$\frac{1}{2b} S_b(x,x,Tx) < S_b(x,x,y) \Rightarrow F\big(2b^3 S_b(Tx,Tx,Ty)\big) \leq F\big(M_T(x,y)\big) - \psi\big(M_T(x,y)\big), \tag{2.1}$$

where $\psi \in \Psi$ and

$$M_T(x,y) = \max\Big\{S_b(x,x,y), \frac{S_b(y,y,Ty)}{10}, \frac{S_b(x,x,Tx)}{10}, S_b(Tx,Tx,Ty), \frac{S_b(y,y,Tx)}{18b}, \frac{S_b(Tx,Tx,T^2x)}{2}\Big\}.$$

Our main result is the following.

Theorem 2.5. [1] *Let (X, S_b) be a complete dislocated S_b-metric space and $T : X \to X$ be a generalized F-Suzuki-contraction satisfying the following condition:*

$$\max\Big\{\frac{S_b(y,y,Ty)}{10}, \frac{S_b(x,x,Tx)}{10}, \frac{S_b(y,y,Ty)}{9} + \frac{S_b(Tx,Tx,Ty)}{18}, \frac{S_b(Tx,Tx,T^2x)}{2}\Big\} \leq S_b(Tx,Tx,Ty)$$

for all x, y in X. Then T has a unique fixed point in X.

Example 2.6. [1] Let $X = \{-1, 0, 1\}$. Define the mapping $S_b : X^3 \to \mathbb{R}_+$ by

$$S_b(x,y,z) = \begin{cases} \frac{3}{2}, & 0 = x = y \neq z = 1 \text{ or } -1 = x = y \neq z = 1 \\ \frac{10}{6}, & 1 = x = y \neq z \\ 0, & x = y = z = -1 \text{ or } 1 \\ \frac{1}{5}, & \textit{otherwise} \end{cases}$$

for all $x, y, z \in X$. Therefore (X, S_b) is a complete dislocated S_b-metric space with $b = \frac{3}{2}$. Put $F(\alpha) = \ln \alpha$ $(\alpha > 0)$ and $\psi(t) = t$ $(t \geq 0)$. Define $T : X \to X$ by

$$T(x) = \begin{cases} 0, & x = 1 \\ -1, & x = -1, 0. \end{cases}$$

Hence, T is a generalized F-Suzuki-contraction which satisfies the assumption of Theorem 2.5 and so it has a unique fixed point -1.

References

[1] H. Mehravaran, M. Khanehgir and R. Allahyari, *Fixed Point Theorems for Generalized F-Contractions and Generalized F-Suzuki-Contractions in Complete Dislocated S_b-Metric Spaces*, Int. J. Anal. Appl., 17 (2019), 734-751.

[2] H. Piri, P. Kumam, *Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces*, Fixed Point Theory Appl., 90 (2016), 13 pages.

[3] Y. Rohená, T. Došenović, S. Radenović, *A note on the paper "A Fixed point Theorems in S_b-Metric Spaces"*, Filomal, 31 11 (2017), 3335–3346.

[4] Sh. Sedghi, A. Gholidahne, T. Došenovic, J. Esfahani, S. Radenovic, *Common fixed point of four maps in S_b-metric spaces*, J. Linear Topol. Alg., 5 2 (2016), 93–104.

[5] Sh. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorem in S-metric spaces*, Mat. Vesn., 64 (2012), 258–266.

[6] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces* Fixed Point Theory Appl., 94 (2012), 6 pages.