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OPTIMAL INVESTMENT FOR INSURERS UNDER CORRELATED LEVY PROCESSES

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Abstract. For the cash flow of an insurance company, insurer invests into a portfolio consisting of risky and riskless assets. We consider optimal investment and risk control problems for an insurer under mean-variance-conditional value at risk criterion. The insurer's risk process is modeled by a Lévy process and it can regulate the risk by controlling the number of insurance policies. Moreover, insurers can invest in financial market consists of one risk-free asset and one risky asset whose price is described by another type of Lévy process. We set a structure that allows a correlation among the risky asset price and risk control process.The explicit expressions for the efficient strategy and efficient frontier derived by martingale approach. In a special case for Gamma processes, we find the conditional value at risk with respect to characteristice functions. Finally a sensitivity analysis is presented to illustrate the effect of parameters on the efficient frontier.

1. Introduction

As the insurance companies have the opportunity to invest in the financial market, there has been an increasing attention in the problem of optimal investment for an insurer. The risk process first was approximated by Brownian motion with drift and stock process is modeled by a geometric Brownian motion. Yang and Zhang [6] extended the optimal investment to jump-diffusion surplus process and Wang [7] considered the case of multiple risky assets. Many empirical investigations have illustrated that the stock price processes have sudden downward (or upward) jumps. This phenomenon can not be accounted by a continuous exponential Brownian motion, so general jump-diffusion or Lévy processes are replaced by the classical Brownian motions. Lévy process is a cádlág stochastic process $(L_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, p)$ with values in $\mathbb{R}^d$ such that $L_0 = 0$ almost surely, it has independent and stationary increments and is stochastically continuous.

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Portfolio selection is to find the asset allocation and stock selection process against the objectives of the fund set. The mean-variance approach by Markowitz provides a fundamental basis for portfolio construction. In this analysis, the expected return is maximized for a given level of risk. Bi et al. [1] studied optimal reinsurance investment strategy for the mean-variance problem in some risk models. The Value-at-Risk (VaR) and the conditional VaR (CVaR) as the risk measures are defined as the quantile of the loss under certain confidence level but it has been widely criticized for the lack of coherent risk measure property. More importantly, the convex of the CVaR measure leads to a tracktable optimization model of the corresponding mean-CVaR portfolio formulation and its extensions [2], [3]. The combined risk measure models as the mean-variance-CVaR (MVC) that reduces both the variance and the CVaR measures has been proposed [5]. One method for solving the optimization problems is Hamilton-Jacobi- Bellman approach. Due to some limitation of it, Martingale method has been introduced and developed to handle problems such as non-Markovian systems. This property shows that the future value of the different investment opportunities only depends on its current value and not their historical values. Among many researches on the dynamic programming to solve the optimization problem, Zou et al. [9] and Zhou et al. [8] investigate on the corresponding optimal control strategies via the Martingale approach.

In this paper, We investigate on the optimal investment and risk regulation for an insurer. We formulate the risky asset and the insurer risk by two stochastic differential equations driven by two correlated pure jump Lévy processes . Moreover, we use MVC criterion in order to make decisions. To find a closed form of the CVaR with respect to characteristic functions, we consider the Gamma processes that models the skewness and kurtosis. In general, an insurer face two risks in his financial activities. A risk arising from risky invesments and a risk process of a insurance company that results from deteministic premium and stochastic claim payments. We assume that trading in the financial market is continuous, without taxes or transaction costs, and all assets are infinitely divisible. Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the filteration and the positive finite constant T represents the terminal time. There are two assets available for investment, one risky asset (stocks and mutual fund) and one riskless asset (bond or bank account). An insurer enters the market with initial wealth x and allocates his wealth in these 2 assets continuously. A riskless asset with price process B and a risky asset (stock) with price process S that are given by $dB(t) = rB(t)dt$ where $r > 0$ is the interest rate and

$$dS(t) = S(t)[bdt + \sigma dL^{(1)}(t)], \tag{1.1}$$

where b is the appreciation rate, $\sigma > 0$ is the volatility, $L^{(1)}(t)$ is a Lévy process. Furthermore, we assume that insurer's risk (per policy) is given by $dR(t) = cdt + d\overline{L}(t)$ where c is a positive constant, $\overline{L}(t)$ is another Lévy process which has correlation with $L^{(1)}(t)$ by ρ. Then, $\overline{L}(t)$ can be written as $\overline{L}(t) = \rho L^{(1)}(t) + \sqrt{1-\rho^2}L^{(2)}(t)$, where $L^{(2)}(t)$ is another Lévy process independent of $L^{(1)}(t)$. Corresponding to a strategy $u(t) := (\pi(t), q(t))$, where $\pi(t)$ is the amount invested in the risky asset, and q(t) is the number of policies (liabilities) at time t, the insurer' s wealth process X with initial capital x follows the dynamics

$$\begin{aligned} dX(t) &= \pi(t)\frac{dS(t)}{S(t)} + [X(t) - \pi(t)]\frac{dB(t)}{B(t)} + pq(t)dt - q(t)dR(t) \\ &= [rX(t) + (b-r)\pi(t) + (p-c)q(t)]dt + [\sigma\pi(t) - \rho q(t)]dL^{(1)}(t) \\ &- \sqrt{1-\rho^2}q(t)dL^{(2)}(t). \end{aligned} \tag{1.2}$$

By solving the SDE , it is verified that $X(t) = e^{rt}(x + Y(t))$ where $Y(t) = \int_0^t e^{-rs}[(b - r)\pi(s) + (p-c)q(s)]ds + \int_0^t e^{-rs}[\sigma\pi(s) - \rho q(s)]dL^{(1)}(s) - \int_0^t e^{-rs}\sqrt{1-\rho^2}q(s)dL^{(2)}(s)$ where p is the average premium per policy for insurer and the conditions $b > r \geq 0$ and $p > c > 0$ on the coefficients hold.

To start the construction of mean-risk portfolio model, we give first the formal definition of the CVaR of the portfolio loss. Let the loss function of the terminal wealth be $F(X(T)) = X(0) - X(T)$. Define the cumulative distribution function of $F(X(T))$ as $\Psi(y) = P(F(X(T)) \leq y)$. For a given confidence level β, the CVaR of the loss function $F(X(T))$ is given as

$$CVaR_\beta[F(X(T))] = \frac{1}{1-\beta}\int_{F(X(T))\geq VaR_\beta} F(X(T))d\Psi_\beta(y).$$

The aim of insurance companies which care about both the variance and CVaR risk measure of the portfolio is to minimize $Var[X(T)] + \omega CVaR_\beta[F(X(T))]$ where the dynamics of $X(t)$ is defined in (1.2), $E[X(T)] = d$ and $X(T) \geq 0$.

2. Optimal portfolio based on martingale approach

The goal is to develop a closed form solution for MVC problem by the martingale approach. So, for solve this optimal problem we should compute the X^{u^*}. Before applying this approach we need some propositions and lemma.

Definition 2.1. A strategy $u(t) := (\pi(t), q(t))$ where $q(t) \geq 0$ is said to be admissible, if $\pi(t)$ and $q(t)$ are progressively measurable and second order integrable. The aim is to maximize the expected terminal utility, i.e, to find an admissible strategy u^* such that $E[U(X^{u^*}(T))] = \sup_{u\in\mathcal{S}} E[U(X^u(T))]$.

In this paper we consider quadratic utility function U as $U(x) = x - \frac{\gamma}{2}x^2$ where $\gamma > 0$ is a parameter and U is strictly concave and continuously differentiable on R. Thus, there exists at most one optimal terminal wealth for insurance company.

Proposition 2.2. *Let $\mathcal{S}$ be the set of all admissible strategies and the utility function is strictly concave and continuously differentiable on $\mathbb{R}$. If there exists a strategy $u^* \in \mathcal{S}$ such that $E[U'(X^{u^*}(T))X(T)]$ is constant over $u \in \mathcal{S}$, then u^* is the optimal strategy.*

The problem of the MVC portfolio selection and risk control for an insurer is to maximize the expected terminal wealth $E[X^{x,u}(T)]$ and to minimize the variance of the terminal wealth $Var[X^u(T)]$ for $u \in \mathcal{S}$ and $CVaR_\beta[F(X(T))]$. This is multi-objective problem with two criteria.

Lemma 2.3. *(Martingale Representation). For any square-integrable martingale* $Z(t) = Z(0) + \sum_{i=1}^{2} \int_0^t \int_R \theta_i(s,x)(\mu_i(ds,dx) - \nu_i(ds,dx))$, *there exists some* $\theta = (\theta_1, \theta_2) \in \theta^2$ *for all* $t \in [0,T]$, *such that* μ_i *denote the jump measure of* $L_t^{(i)} = \int_0^t \int_R x(\mu_i(ds,dx) - \nu_i(ds,dx))$ *and* ν_i *denote the dual predictable projection of* μ_i *which has the form* $\nu_i(dt,dx) = dt \times m_i(dx)$ *with* $m_i(0) = 0$ *and* $\int_R (x^2 \wedge 1) m_i(dx) < \infty$. *Throughout this work, we assume that* $M_i := \int_R x^2 m_i(dx) < \infty$.

Theorem 2.4. *Let* $u^* \in \mathcal{S}$, *then* u^* *satisfies proposition 2.2 if and only if there exists a* (θ_1, θ_2) *such that* $(X^{u^*}, u^*, Z^*, \theta_1^*, \theta_2^*)$ *solves the optimization problem by the dynamics of* $X(t)$ *and* $Z(t)$ *where* $X(0) = x$ *and* $Z(T) = 1 - \gamma X(T)$. *So we have* $X^{u^*}(T) = \frac{1}{\gamma} - \frac{Z(0)}{\gamma A(T)} - \frac{1}{\gamma A(T)} \int_0^T Z(t) a(t) A(t) dt - \sum_{i=1}^{2} \frac{1}{\gamma A(T)} \int_0^T \int_R A(t) \theta_i(t,x)(\mu_i(dt,dx) - \nu_i(dt,dx))$, *where* $q^*(t) = e^{-r(T-t)} \frac{\rho(b-r)+\sigma(p-c)}{\sigma M_2(1-\rho^2)} \frac{A(t)}{\gamma A(T)} Z(t)$ *and* $\pi^*(t) = e^{-r(T-t)} \frac{(b-r)(M_2 - \rho^2 M_2 + \rho^2 M_1) + \rho M_1 \sigma (p-c)}{\sigma^2 M_1 M_2 (1-\rho^2)} \frac{A(t)}{\gamma A(T)} Z(t)$ *in which* $\theta_1^*(t,x) = -x \frac{\rho(b-r)(M_2 - \rho^2 M_2)}{\sigma M_1 M_2 (1-\rho^2)} Z(t)$ *and* $\theta_2^*(t,x) = x \frac{\rho(b-r)+\sigma(p-c)}{\sigma M_2 \sqrt{1-\rho}} Z(t)$.

It is well-known that finding a mean-variance efficient strategy is equivalent to maximizing the expected quadratic utility. Since Z is a square-integrable martingale and $X^{u^*}(T) = \frac{1-Z(T)}{\gamma}$, we obtain $E[X^{u^*}(T)] = \frac{1-Z(0)}{\gamma}$, $Var[X^{u^*}(T)] = \frac{1}{\gamma^2}(E[Z(T)^2] - Z(0)^2)$ and $CVaR_\beta[F(X^{u^*}(T)]$. For finding the $CVaR$ we have to provide the density function of $X^{u^*}(T)$ as Gamma processes with $L^{(1)}(t; \delta_1, \vartheta_1, \eta_1)$ and $L^{(2)}(t; \delta_2, \vartheta_2, \eta_2)$ where (i) δ the volatility of the Brownian process ,(ii) ϑ the variance rate of gamma time change and (iii) η the drift in the Brownian process process with drift.
Finally in sensitivity analysis, first we provide some numerical simulation and illustrations then show the impact of the market parameters on the efficient frontier under the criterion of MVC.

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