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A FILMILY OF MEASURES OF NONCOMPACTNESS IN THE HÖLDER SPACE $C^{n,\gamma}(\mathbb{R}_+)$

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Abstract. In this paper, we prove the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t,u(t)), \ \alpha \in (n,n+1), \ 0 \le t < +\infty, \\ u(0) = 0, \ u''(0) = 0, \dots, u^{(n)}(0) = 0, \ \lim\limits_{t\to+\infty} {}^cD^{\alpha-1}u(t) = \beta u(\xi). \end{cases}$$

The considerations of this paper are based on the concept of a new family of measures of noncompactness in the space of functions $C^{n,\gamma}(\mathbb{R}_+)$ and a fixed point theorem of Darbo type. We also provide an illustrative example in support of our existence theorems.

1. Introduction and Preliminaries

The fractional calculus, an active branch of mathematics analysis, is as old as the classical calculus which we know today. Fractional differential equations arise in the fields of physics, chemistry, engineering, and economics, etc., (see [4]). Also, some basic theory for the initial value problems of fractional differential operators has been discussed in [5] and some other boundary value problems of fractional type have been studied in [3]. On the other hand, a more effective approach consists in applying new and more suitable regular measures of noncompactness for some Fréchet spaces and Banach spaces defined on a bounded or an unbounded interval.

In this paper, we prove the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t,u(t)), \ \alpha \in (n,n+1), \ 0 \le t < +\infty, \\ u(0) = 0, \ u''(0) = 0, \dots, u^{(n)}(0) = 0, \ \lim\limits_{t\to+\infty} {}^cD^{\alpha-1}u(t) = \beta u(\xi). \end{cases} \tag{1.1}$$

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The considerations of this paper are based on the concept of a new family of measures of noncompactness in the space of functions $C^{n,\gamma}(\mathbb{R}_+)$ satisfying the Hölder condition and a fixed point theorem of Darbo type.

Let E be a Banach space with zero element 0. For a nonempty subset X of E, the symbols $\overline{X}$ and $\mathrm{Conv}X$ will denote the closure and closed convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.
Let $n \in \mathbb{N}$, we denote by $C^n(\mathbb{R}_+)$ the space of all real functions which are n-times continuously differentiable on $\mathbb{R}_+$ with the family of seminorms

$$|x|^T_{C^n} = \sup\{|x^{(k)}(t)| : \ 0 \le k \le n, \ t \in [0,T]\}$$

for $T \ge 1$. The space $C^n(\mathbb{R}_+)$ would be a Fréchet space furnished with the distance

$$d(x,y) = \sup\big\{\frac{1}{2^T}\min\{1, |x-y|^T_{C^n}\} : T \in \mathbb{N}\big\}.$$

A nonempty subset $X \subset C^n(\mathbb{R}_+)$ is said to be bounded if $\sup\{|x|^T_{C^n} : x \in X\} < \infty$ for all $T \in \mathbb{N}$. For $\gamma \in (0,1]$; the space γ-Hölder continuous functions $\mathcal{H}_\gamma(\mathbb{R}_+)$ is the family of all continuous functions $x = x(t)$ on $\mathbb{R}_+$, where

$$\forall T > 0 \ \exists M_T > 0, \ \sup\big\{\frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\gamma} : \ t_1, t_2 \in [0,T], \ t_1 \neq t_2\big\} < M_T.$$

The space $\mathcal{H}_\gamma(\mathbb{R}_+)$ is equipped with the family of seminorms

$$|x|^T_{\mathcal{H}_\gamma} = |x(0)| + \sup\big\{\frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\gamma} : \ t_1, t_2 \in [0,T], \ t_1 \neq t_2\big\}.$$

$\mathcal{H}_\gamma(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$d(x,y) = \sup\big\{\frac{1}{2^T}\min\{1, |x-y|^T_{\mathcal{H}_\gamma}\} : T \in \mathbb{N}\big\}.$$

Definition 1.1. ([4]) For at least n-times continuously differentiable function $f : [0,\infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 1.2. [1] *Let $f \in L^1([0,\infty))$ be a nonnegative continuous function, $n < \alpha < n+1$, and β and ξ be positive real numbers. Then the boundary value problem of fractional differential equation (1.1) has a unique solution*

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds + \frac{t}{\beta\xi}\int_0^\infty f(s)ds - \frac{t}{\xi\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}f(s)ds.$$

Theorem 1.3. *(Darbo [2]) Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : C \to C$ be a continuous mapping. Assume that a constant $k \in [0,1)$ exists such that $\mu(F(X)) \le k\mu(X)$, for any nonempty subset X of C, where μ is a measure of noncompactness defined in E. Then F has a fixed point in the set C.*

2. Main results

For $0 < \gamma \leq 1$ fixed, the space $C^{n,\gamma}(\mathbb{R}_+)$ consists of all functions $u \in C^n(\mathbb{R}_+)$ whose n^{th}-derivative is Hölder continuous with exponent γ. Then $C^{n,\gamma}(\mathbb{R}_+)$ equipped with the family of seminorms

$$|u|_\gamma^T = |u|_{C^n}^T + |u^{(n)}|_{\mathcal{H}_\gamma}^T$$

for each $T \in \mathbb{N}$. $C^{n,\gamma}(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$d(x,y) = \sup\big\{\frac{1}{2^T}\min\{1, |x-y|_\gamma^T\} : T \in \mathbb{N}\big\}.$$

Theorem 2.1. [1] *A subset $\mathcal{F} \subset C^{n,\gamma}(\mathbb{R}_+)$ is totally bounded (relatively compact) if and only if satisfies the following conditions:*

(i) *The set $\mathcal{F}^{(k)} = \{f^{(k)} : f \in \mathcal{F}\}$ is bounded and equicontinuous on $[0,T]$ for all $0 \leq k \leq n$ and $T \in \mathbb{N}$ respect to C^n seminorms.*

(ii) *The set $\mathcal{F}^{(n)} = \{f^{(n)} : f \in \mathcal{F}\}$ is bounded and equicontinuous on $[0,T]$ for all $T \in \mathbb{N}$ with respect to the modulus of continuity function $w(r) = r^\gamma$, which means that*

$$\forall \varepsilon > 0\ \forall T > 0\ \exists \delta_T > 0\ \forall f \in \mathcal{F}\ \forall t_1, t_2 \in [0,T],\ \big(|t_1 - t_2| < \delta_T \Rightarrow \frac{|f^{(n)}(t_1) - f^{(n)}(t_2)|}{|t_1 - t_2|^\gamma} \leq \varepsilon\big).$$

Definition 2.2. [1] A family of mappings $\{\vartheta_{\gamma,n}^T\}_{T\in\mathbb{N}}$, $\vartheta_{\gamma,n}^T : \mathfrak{M}_{C^{n,\gamma}} \to \mathbb{R}_+$, where $\gamma \in (0,1]$ is said to be a family of measures of noncompactness in $C^{n,\gamma}(\mathbb{R}_+)$ if it fulfils the following conditions:

1° The family $\ker\{\vartheta_{\gamma,n}^T\} = \{X \in \mathfrak{M}_{C^{n,\gamma}} : \vartheta_{\gamma,n}^T(X) = 0$ for each $T \in \mathbb{N}\}$ is nonempty and $\ker\{\vartheta_{\gamma,n}^T\} \subseteq \mathfrak{N}_{C^{n,\gamma}}$.
2° $U \subset V$ implies that $\vartheta_{\gamma,n}^T(U) \leq \vartheta_{\gamma,n}^T(V)$ for $T \in \mathbb{N}$.
3° $\vartheta_{\gamma,n}^T(\overline{U}) = \vartheta_{\gamma,n}^T(U)$ for $T \in \mathbb{N}$.
4° $\vartheta_{\gamma,n}^T(\mathrm{Conv}U) = \vartheta_{\gamma,n}(U)$ for $T \in \mathbb{N}$.
5° $\vartheta_{\gamma,n}^T(\lambda U + (1-\lambda)V) \leq \lambda\vartheta_{\gamma,n}^T(U) + (1-\lambda)\vartheta_{\gamma,n}^T(V)$ for $\lambda \in [0,1]$, and $T \in \mathbb{N}$.
6° If $\{U_j\}$ is a sequence of closed chains of $\mathfrak{M}_{C^{n,\gamma}}$ such that $U_{j+1} \subset U_j$ for $j = 1, 2, \ldots$ and if $\lim_{j\to\infty} \vartheta_{\gamma,n}^T(U_j) = 0$ for each $T \in \mathbb{N}$, then the intersection set $U_\infty = \bigcap_{j=1}^{\infty} U_j$ is nonempty.

Suppose $T \in \mathbb{N}$ and U is a bounded subset of the space $C^{n,\gamma}(\mathbb{R}_+)$. For $u \in U$, and $\varepsilon > 0$ set the following quantities,

$$\omega^T(u,\varepsilon) = \sup\{|u^{(k)}(t_1) - u^{(k)}(t_2)| : t_1, t_2 \in [0,T], |t_1 - t_2| \leq \varepsilon,\ k = 0, 1, \ldots, n\},$$

$$\mu_\gamma^T(u,\varepsilon) = \sup\big\{\frac{|u^{(n)}(t_1) - u^{(n)}(t_2)|}{|t_1 - t_2|^\gamma} :\ t_1, t_2 \in [0,T],\ t_1 \neq t_2,\ |t_1 - t_2| \leq \varepsilon\big\},$$

$\omega^T(U,\varepsilon) = \sup_{u\in U}\omega^T(u,\varepsilon)$, $\mu_\gamma^T(U,\varepsilon) = \sup_{u\in U}\mu_\gamma^T(u,\varepsilon)$ and $\vartheta_{\gamma,n}^T(U,\varepsilon) := \omega^T(U,\varepsilon) + \mu_\gamma^T(U,\varepsilon)$.

It is easy to see that the function $\varepsilon \to \vartheta_{\gamma,n}^T(U,\varepsilon)$ is increasing on the interval $(0,\infty)$, so the $\lim_{\varepsilon\to 0}\vartheta_{\gamma,n}^T(U,\varepsilon)$ exists and we have the following result expressing the family of measures of noncompactness in the space $C^{n,\gamma}(\mathbb{R}_+)$.

Theorem 2.3. [1] *The family of functions $\{\vartheta_{\gamma,n}^T\}_{T\in\mathbb{N}}$, $\vartheta_{\gamma,n}^T : \mathfrak{M}_{C^{n,\gamma}} \to \mathbb{R}_+$ given by $\vartheta_{\gamma,n}^T(U) = \lim_{\varepsilon\to 0}\vartheta_{\gamma,n}^T(U,\varepsilon)$, defines a regular family of measures of noncompactness on $C^{n,\gamma}(\mathbb{R}_+)$.*

The following theorem is a version of Darbo's fixed point theorem in $C^{\gamma,n}(\mathbb{R}_+)$.

Theorem 2.4. [1] *Let C be a nonempty, closed and convex subset of the Fréchet space $C^{\gamma,n}(\mathbb{R}_+)$ and $F : C \to C$ be a continuous operator such that for each $T \in \mathbb{N}$ there exists $L_T \in [0,1)$ so that $\vartheta^T_{\gamma,n}(FX) \leq L_T \vartheta^T_{\gamma,n}(X)$ for each $X \subset C$. Then F has at least one fixed point in the set C.*

3. Application

We will assume that the following conditions are satisfied.
(i) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function, such that there exist increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \to 0$, and $\theta(t) \to 0$ as $t \to 0$, $\varphi \in L^1([0,\infty))$ and the inequalities

$$|f(s,u) - f(s,v)| \leq \varphi(|u-v|), \ \int_0^\infty |f(s,u) - f(s,v)| ds \leq M\theta(|u-v|),$$

hold for all $s \in \mathbb{R}_+$, $u, v \in \mathbb{R}$ and for some $M > 0$. We also suppose that

$$\overline{N} = \sup\{|f(s,0)| : \ s \in \mathbb{R}_+\} < \infty, \text{ and } \overline{G} = \int_0^\infty |f(s,0)| ds < \infty.$$

Moreover, we assume that
(ii) for each $T \in \mathbb{N}$, $r(T) > 0$ exists which is a solution of the inequality

$$(\varphi(r(T)) + \overline{N})\left(d_T + \frac{T\xi^{\alpha-1}}{\alpha\Gamma(\alpha)} + \frac{2T^{\alpha-n-\gamma}}{(\alpha-n)\Gamma(\alpha-n)}\right) + \frac{T}{\beta\xi}(M\theta(r(T)) + \overline{G}) \leq r(T),$$

where $d_T = \sup\{\frac{T^{\alpha-k}}{(\alpha-k)\Gamma(\alpha-k)}, \ k = 0, 1, 2, \ldots, n\}$.

Theorem 3.1. [1] *Suppose that the assumptions (i) and (ii) are satisfied. Then the nonlinear Caputo fractional differential equation (1.1) has at least one solution in the space $C^{n,\gamma}(\mathbb{R}_+)$.*

Example 3.2. [1] Consider the following fractional differential equation

$$\begin{cases} {}^cD^{\frac{5}{2}}u(t) = \frac{e^{-t}\arctan(u(t)+1)}{\sqrt{t+10}}, \\ u(0) = 0, \ u''(0) = 0, \ \lim\limits_{t\to+\infty} {}^cD^{\frac{3}{2}}u(t) = 6u(\frac{1}{2}), \end{cases} \tag{3.1}$$

Observe that Eq. (3.1) is a special case of the Eq. (1.1), so (3.1) has at least one solution which belongs to the space $C^{2,\frac{1}{3}}(\mathbb{R}_+)$.

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