National Seminar on Control and Optimization
Nov 13-14, 2019
3th
سومین سمینار ملی
کنترل و بهینه‌سازی
۲۲-۲۳ آبان ۱۳۹۸

SOLUTION OF INTEGRAL EQUATIONS OF NONLINEAR CONVOLUTION TYPE

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Abstract. In this paper, first we introduce a new measure of noncompactness in Weighted Sobolev space $W_w^{m,p}(\Omega)$ $(\Omega \subseteq \mathbb{R}^{\mathbb{N}})$ and then, as an application, we study the existence of solutions for a class of the nonlinear convolution type integral equations by using Darbo's fixed point theorem associated with this new measure of noncompactness.

1. Introduction

Sobolev spaces [2], i.e., the class of functions with derivatives in L^p, play an outstanding role in the modern analysis. In the last decades, there has been increasing attempts to study of these spaces. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces [5, 6]. On the other hand, integral-differential equations (IDE) have a great deal of applications in some branches of sciences. It arises especially in a variety of models from applied mathematics, biological science, physics and another phenomenon, such as the theory of electrodynamics, electromagnetic, fluid dynamics, heat and oscillating magnetic. Sobolev spaces without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. The most important application of measures of noncompactness in the fixed point theory is contained in the Darbos fixed point theorem [3, 4, 6].

2. Measures of noncompactness on weighted Sobolev spaces $W_w^{m,p}(\Omega)$

Definition 2.1. [1] A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

2010 *Mathematics Subject Classification.* Primary: 47H08; Secondary: , 47H10, 44A35.
Key words and phrases. Measures of noncompactness, Darbo's fixed point theorem, Convolution differential equations, Weighted Sobolev spaces.

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1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;
2° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;
3° $\mu(\overline{X}) = \mu(X)$;
4° $\mu(Conv X) = \mu(X)$;
5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

Theorem 2.2. [1] *Let Ω be a nonempty, bounded, closed, and convex subset of a space E and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property*

$$\mu(FX) \leq k\mu(X), \tag{2.1}$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

Definition 2.3. [2] A weight on $\mathbb{R}^N$ is a locally integrable function w such that $w(x) > 0$ for *a.e.* $x \in \mathbb{R}^N$.

Every weight w gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will also be denoted by w. Thus, $w(E) = \int_E w dx$ for measurable sets $E \subset \mathbb{R}^N$. Let w be a weight, and let $\Omega \subseteq \mathbb{R}^N$ be open. For $0 < p < \infty$, we define $L^p_w(\Omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p_w(\Omega)} = \Big(\int_\Omega |f(x)|^p w dx\Big)^{\frac{1}{p}} < \infty.$$

Definition 2.4. [2] Let $\Omega \subseteq \mathbb{R}^N$ be open, $1 \leq p < \infty$, and m a positive integer. Suppose that the weight $w \in A_p$. Then we define the weighted Sobolev space $W^{m,p}_w(\Omega)$ as the set of functions $u \in L^p_w(\Omega)$ with weak derivatives $D^\alpha u \in L^p_w(\Omega)$ for $|\alpha| \leq m$. The norm of u in $W^{m,p}_w(\Omega)$ is given by

$$\|u\|_{W^{m,p}_w(\Omega)} = \big(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^p w dx\big)^{\frac{1}{p}}.$$

When $w = 1$, these spaces will be denoted $W^{m,p}(\Omega)$.

Before introducing the new measures of noncompactness on the spaces $W^{m,p}_w(\Omega)$, we need to characterize the compact subsets of $W^{m,p}_w(\Omega)$.

Theorem 2.5. [2] *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^N)$ and $1 \leq p < \infty$. Then $\mathcal{F}$ is relatively compact if and only if the following conditions are satisfied:*

(*i*) $\lim_{h\longrightarrow 0} \|\tau_h f - f\|_p = 0$ *uniformly with respect to $f \in \mathcal{F}$, where $\tau_h f(x) = f(x+h)$ for $x, h \in \mathbb{R}^N$.*

(*ii*) *For $\varepsilon > 0$ there exists a bounded and measurable subset $\Omega \subset \mathbb{R}^{\mathbb{N}}$ such that*

$$\|f\|_{L^p(\mathbb{R}^{\mathbb{N}}\setminus\Omega)} < \varepsilon,$$

for $f \in \mathcal{F}$.

Let U be a bounded subset of the space $W_w^{m,p}(\Omega)$. For $u \in U$, and $\varepsilon > 0$. Let us denote

$$\begin{aligned}
&\omega(u,\varepsilon) = \sup\{\|u^{(k)}(t+h) - u^{(k)}(t)\|_{W_w^{m,p}(B_T)},\ \|h\|_{\mathbb{R}^N} < \varepsilon,\ 0 \le k \le m\},\\
&\omega(U,\varepsilon) = \sup\{\omega(u,\varepsilon) : u \in U\},\\
&\omega(U) = \lim_{\varepsilon\to 0} \omega(U,\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&d_T(U) = \sup\{\|u^{(k)}\|_{W_w^{m,p}(\Omega\setminus B_T)} :\ u \in U,\ 0 \le k \le m\},\\
&d(U) = \lim_{T\to\infty} d_T(U),\\
&\omega_0(U) = \omega(U) + d(U).
\end{aligned}$$

We have the following fact.

Theorem 2.6. *The function* $\omega_0 : \mathfrak{M}_{W_w^{m,p}(\Omega)} \to \mathbb{R}_+$ *is a measure of noncompactness on weighted Sobolev space* $W_w^{m,p}(\Omega)$ *and moreover,* $\ker\omega_0 = \mathfrak{N}_{W_w^{m,p}(\Omega)}$.

3. Application

In this section we show the applicability of our results and study the existence of solutions for functional integral equation of convolution type. Further, we present an illustrative example to verify the effectiveness and applicability of our results.

Definition 3.1. [1] A function $f : \mathbb{R}^N \times \mathbb{R}^M \longrightarrow \mathbb{R}$ is said to have the Carathéodory property if

- (i) the function $t \to f(t,u)$ is measurable for any $u \in \mathbb{R}^M$.
- (ii) the function $u \to f(t,u)$ is continuous for almost all $t \in \mathbb{R}^N$.

Theorem 3.2. *Assume that the following conditions are satisfied:*
(i) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions,* $\frac{df}{dx_i}$ $(i = 1,2,\ldots,N)$ *are continuous and there are exists a constant* $\lambda_1, \lambda_2 \in [0,1)$ *and* $a, b \in W_w^{1,p}(\Omega)$ *such that*
$|f(x,u) - f(y,v)| \le |a(x) - a(y)| + \lambda_1|u - v|$, $\quad |\frac{df}{dx_i}(x,u) - \frac{df}{dy_i}(y,v)| \le |a(x) - a(y)| + \lambda_1|u - v|$,
$|\frac{df}{du}(x,u)\frac{du}{dx_i}(x)| \le \lambda_2|\frac{du}{dx_i}(x)|$, $\quad |\frac{df}{du}(x,u)\frac{du}{dx_i}(x) - \frac{df}{dv}(y,v)\frac{dv}{dy_i}(y)| \le M|b(x) - b(y)| + \lambda_2|\frac{du}{dx_i} - \frac{dv}{dy_i}|$.
for any $u, v \in \mathbb{R}$ *and almost all* $x, y \in \Omega \subseteq \mathbb{R}^N$, $0 < M < \infty$ *and* $\lambda_1 + \lambda_2 < 1$.
(ii) $f(x,0) \in W_w^{1,p}(\Omega)$.
(iii) $G : \Omega \times \Omega \to \mathbb{R}$ *satisfies the Carathéodory conditions,* $\frac{dG}{dx_i}$ $(i = 1,2,\ldots,N)$ *are continuous and there exists* $g_1, g_3 \in W_w^{1,p}(\Omega)$ *and* $g_2 \in L^\infty(\Omega)$ *such that for all* $x, y \in \mathbb{R}^N$ *we have*
$|G(x,y)| \le g_1(x)g_2(y)$, $\quad |\frac{dG}{dx}(x,y)| \le g_1(x)g_2(y)$,
$|G(x_1,y) - G(x_2,y)| \le |g_3(x_1) - g_3(x_2)||g_2(y)|$, $\quad |\frac{dG}{dx_i}(x_1,y) - \frac{dG}{dx_i}(x_2,y)| \le |g_3(x_1) - g_3(x_2)||g_2(y)|$.
(iv) *The operator* Q *acts continuously from the space* $W_w^{1,p}(\Omega)$ *into itself and there exists a increasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|Qu\|_{W_w^{1,p}(\Omega)} \le \psi(\|u\|_{W_w^{1,p}(\Omega)}),$$

for any $u \in W^{1,p}_w(\Omega)$.

(*v*) *There exists a positive solution* r_0 *to the inequality*

$$(\lambda_1 + \lambda_2)r + \|f(x,0)\|_{W^{1,p}_w(\Omega)} + \|g_1\|_{W^{1,p}_w(\Omega)}\|g_2\|_{L^\infty}\psi(r) \leq r.$$

Then the functional integral equation of convolution type

$$u(x) = f(x,u(x)) + \int_{\mathbb{R}^N} G(x,y)(Qu)(y)dy,$$

has at least one solution in the space $W^{1,p}_w(\Omega)$.

Example 3.3. Consider the following functional integral equation

$$u(x) = \frac{\arctan u(x)}{\sqrt{x+7}} + \int_0^1\int_0^1\int_0^1 \frac{e^{-(x_1^2+2y_2+\sqrt[4]{y_3}+3)}}{(y_1+2)^2(y_2+2)^2(y_3+6)}\ln(1+|u(y)|)dy_1dy_2dy_3, \tag{3.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We study the solvability of integral equation (3.1) on the space $W^{1,p}_w(\Omega)$, with weight $w = 1$. Observe that Eq. (3.1) is a special case of the Eq. (1.1) when

$$\begin{aligned}
f(x,u) &= \frac{\arctan u(x)}{\sqrt{x+7}},\\
G(x,y) &= \frac{e^{-(x_1^2+2y_2+\sqrt[4]{y_3})}}{(y_1+2)^2(y_2+2)^2(y_3+6)}, \quad \big(\, x = (x_1,x_2,x_3)\big)\\
(Qu)(y) &= \ln(1+|u(y)|),\\
g_1(x_1,x_2,x_3) &= g_3(x_1,x_2,x_3) = e^{-(x_1^2+2y_2+\sqrt[4]{y_3}+3)},\\
g_2(x_1,x_2,x_3) &= \frac{1}{(y_1+2)^2(y_2+2)^2(y_3+6)}.
\end{aligned}$$

References

[1] Aghajani, A., O'Regan, D., Shole Haghighi, A.: Measure of noncompactness on $L^p(\mathbb{R}^n)$ and applications. Cubo A Math. J. **17**, 85–97 (2015).

[2] Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. Springer Science, Business Media, LLC (2011).

[3] Kayvanloo, H. A., Khanehgir, M., and Allahyari, R. . A family of measures of noncompactness in the space $\boldsymbol{L^p_{loc}}(\mathbb{R}^{\boldsymbol{N}})$ and its application to some nonlinear convolution type integral equations. Cogent Math. Stat. **6** (1),(2019) Art. ID 1592276, 13 pp.

[4] Kayvanloo, H. A., Khanehgir, M., and Allahyari, R. . A family of measures of noncompactness in the Hölder space $C^{n,\gamma}(\mathbb{R}_+)$ and its application to some fractional differential equations and numerical methods. J. Comput. Appl. Math. 363 (2020) 256–272.

[5] Khanehgir, M., Allahyari, R., and Gholamian, N., Construction of a measure of noncompactness in Sobolev spaces with an application to functional integral-differential equations. Math. Sci. **12** (2018) 1724.

[6] M.H.Z. Moghaddam, R. Allahyari, M.E. Omidvar, A. Shole Haghighi, Existence of solutions for some classes of integro-differential equations in the Sobolev space $W^{n,p}(\mathbb{R}_+)$, J. Fixed Point Theory .Appl. **20**(3)(2018) 99.