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SOLUTION OF OPTIMAL CONTROL VAN DER POL PROBLEM USING MULTIPLE SHOOTING ALGORITHM

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Abstract. In this work, we propose a numerical algorithm for solving optimal control van der pol problem. This approach is based multiple shooting method. A numerical simulation demonstrates the control performance and the stability of the proposed method.

1. Introduction

There are at least two basically different ways of solving optimal control problems. In the indirect approach, the controls are expressed by the maximum principle in terms of state and adjoint variables, which can be computed by solving a possibly very intricate multipoint boundary value problem with jumps and switching conditions. In recent years reliable, stable and efficient numerical algorithms have been developed for the solution of this general class of problems, based on the multiple shooting technique , which actually made accessible the wide applicability of the indirect approach, which includes control- or state-constrained and Chebyshev- roblems as well as feed-back control.
Multiple shooting is one of many methods for finding an optimal trajectory. A trajectory is a full description of the path that a dynamical system can take between two points in state space. An optimal trajectory is a trajectory that minimizes some cost function[1]. The present paper introduces a numerical algorithm for the direct approach, which solves the

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optimal control problem directly in terms of control and state variables. In the new method, a multiple shooting parameterization of the state differential equations is coupled with a simultaneous control parameterization. This leads to a large constrained finite optimization problem, for which a specially suited recursive quadratic programming algorithm with new high rank update formulae is developed, that leads to a substantical improvement of performance compared to previous direct approaches. The algorithm is globally convergent, its local convergence is super-linear with an asymptotic convergence rate that is essentially independent of the mesh size used in the parameterization. In view of practical applications, it is one of the most important properties of the new algorithm that it is completely derivative-free (due to internal numerical differentiation schemes). This means, that any analytical preparations (such as derivation of adjoint equations) are strictly avoided.

2. Optimal Control Problem formulation

We consider the following general class of optimal control problems (OCPs)

$$\begin{aligned} &\min J(x(.), u(.)) = \int_{t_0}^{t_f} f(t, x(t), u(t))dt \\ &s.t.\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \forall t \in \tau \\ &r(t_i, x(t_i) = 0 \quad 0 \le i \le m \\ &g(t_i, u(t_i) \ge 0, \quad 0 \le i \le m, \end{aligned} \tag{2.1}$$

which we strive to minimize the objective function $J(.)$ depending on the trajectory $x(t) \in \mathbb{R}^n$ of a dynamic process described in terms of a system $f : \tau \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ of (ODEs), running on a time horizon $\tau := [t_0, t_f] \subset \mathbb{R}$ and governed by a control trajectory $u(t) \in \mathbb{R}^n$ subject to optimization. The process trajectory $x(.)$ and the control trajectory $u(.)$ shall satisfy certain inequality path constraints $r : \tau \times \mathbb{R}^n \to \mathbb{R}^n$ and $g := \tau \times \mathbb{R}^n \to \mathbb{R}^n$ on a prescribed grid on τ consisting of $m + 1$ grid points

$$t_0 < t_1 < \cdots < t_m = t_f \quad m \in \mathbb{N} \tag{2.2}$$

In order to make this infinite-dimensional OCPs computationally accessible, the direct multiple shooting method is applied to discretize the control trajectory $u(.)$ subject to optimization.

2.1. **Parmeterization of OCPs by Multiple Shooting.** We introduce a discretization of the control trajectory $u(.)$ by defining a shooting grid (2.2) which shall be a super-set of the constraint grid used in (2.1). For clarity, we assume in the following that the two grids coincide, though this is not a theoretical or algorithmic requirement. On each interval of the shooting grid (2.2), we introduce a vector $q_i \in \mathbb{R}^n$ of control parameters together with an associated control base function $b_i = \tau \times \mathbb{R}^n \to \mathbb{R}^n$ with local support,

$$u(t) = \sum_{j=1}^{n_i} b_{ij}(t, q_{ij}), \quad \tau \in [t_i, t_{i+1}] \subseteq \tau, \quad 0 \le i \le m-1 \tag{2.3}$$

The number and location of the shooting grid points and the choice of base functions obviously affect the approximation quality of the optimal solution of the discretized problem.

In addition, we introduce state vectors $s_i \in \mathbb{R}^n$ in all shooting nodes serving as initial values for m initial value problems

$$\dot{x}_i = f(t, x_i(t), q_i), \quad x_i(t_i) = s_i, \quad t \in [t_i, t_{i+1}] \subseteq \tau, \quad 0 \le i \le m-1 \tag{2.4}$$

This parametrization of the process trajectory $x(.)$ will in general be discontinuous on τ.

Continuity of the solution is ensured by introduction of additional matching conditions

$$x_i(t_{i+1}; t_i, s_i, q_i) - s_{i+1} = 0, \quad 0 \le i \le m-1 \tag{2.5}$$

where $x_i(t_{i+1}; t_i, s_i, q_i)$ denote the state trajectory's value $x_i(.)$ in t_{i+1} depending on the start time t_i , initial value s_i , and control parameters q_i on that interval.

The path constraints of problem (2.1) are enforced on the nodes of the shooting grid (2.2) only. While in general it can be observed that this formulation already leads to a solution that satisfies the path constraints on the whole of τ . To ensure this in a rigorous fashion. For clarity, we define the combined constraint functions $r : \tau \times \mathbb{R}^n \to \mathbb{R}^n$

$$0 \le r_i^*(t_i, s_i, q_i), \quad 0 \le i \le m-1, \quad 0 \le r_m(t_m, s_m) \tag{2.6}$$

with $n_i^{r^*} = n_i^r + n_i^g$. These comprise all discretized path constraints as well as equality and inequality point constraints.

The objective function $J(x(.), u(.))$ shall be separable with respect to the shooting grid structure. In general, $J(.)$ will be a Mayer-type function evaluated at the end of the horizon τ or Lagrange type integral objective evaluated on the whole of τ . For both types, a separable formulation is easily found,

$$J(x(.), u(.)) = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} l_i(x_i(t), q_i) dt \tag{2.7}$$

Summarizing, the discretized multiple shooting optimal control problem can be cast as a nonlinear problem

$$\begin{aligned} &\min_{\omega} \ \sum_{i=0}^{m} l_i(\omega_i) \\ &x_i(t_{i+1}; t_i, \omega_i) - s_{i+1} = 0 \\ &r_i(\omega_i) \ge 0 \end{aligned} \tag{2.8}$$

with the vector of unknowns ω being

$$\begin{aligned} &\omega = (s_1, q_1, \cdots, s_{m-1}, q_{m-1}, s_m) \\ &\omega_i = (s_i, q_i), \quad 0 \le i \le m-1, \quad \omega_m := s_m \end{aligned}$$

where the evaluation of the matching condition constraint (2.8) requires the solution of an initial value problem (2.4) [2].

3. Numerical Results

The Van der Pol (VDP) Problem is relatively simple and was chosen for comparison purposes [4]

$$
\begin{aligned}
&Min\ \frac{1}{2}\int_0^{t_f}(x_1^2+x_2^2+u^2)dt\\
&\dot{x}_1 = x_2,\quad x_1(0)=1,\quad x_1(t)=1,\quad t\in[0,t_f]\\
&\dot{x}_2 = -x_1+(1-x_1^2)x_2+u,\quad x_2(0)=0,\quad x_2(t)=0,\ t\in[0,t_f]\\
&u(t)\in[0,2],\quad t\in[0,t_f]\\
&x_1(5)-x_2(5)+1=0
\end{aligned}
\tag{3.1}
$$

Table 1 shows the results of ICLOCS package for a piecewise linear control parameterization. The initial guesses were $x_1(t_j)=1$, $x_2(t_j)=0$. Also, in Table 1 quotes the results

Table 1. Numerical results for (3.1).

Methods	ICLOCS	[3]
Number of function evaluations	56	36
Number of qradient evaluations (iterations)	9	31
CPU-sec total	1.53	15.02
$J(x,u)$	1.6875	1.6875

of Pouliot et al. [3] obtained by a single shooting nonlinear programming algorithm with the same control parameterization and same initial data. The graphs of x_1, x_2 and u are shown in figure 1. This results show the accuracy of our method in comparison with [3].

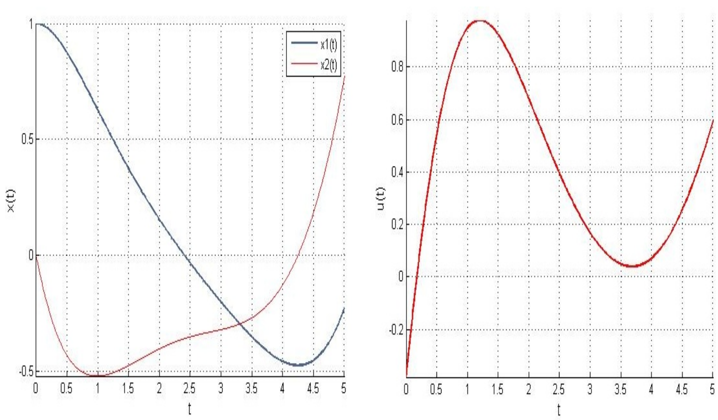

Figure 1. VDP: state variable (left) and control u (right)

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