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OBSERVER DESIGN FOR WAVE PDE WITH NONLINEAR BOUNDARY CONDITION

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Abstract. This paper presents the observer design of one-dimensional wave equation with control at one end and nonlinear boundary condition at another end. First, an exponential observer is designed. Next, an observer-based output feedback control which causes the real system to be exponential stable is proposed. Finally, the simulation results are given to verify the effectiveness of the proposed technique.

1. Introduction

Many major engineering and industrial processes are governed by partial differential equation (PDEs). Recently, stabilization of the wave PDEs have been received a considerable attention [1, 2, 3]. However, as far as we know there are just some papers that take into account the stability of nonlinear wave PDE. In this paper, we consider the observer design problem for the one-dimensional wave equation associated with the cubic nonlinear

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boundary condition in the form

$$\begin{cases} w_{xx}(x,t) = w_{tt}(x,t), & x \in (0,1),\ t > 0, \\ w_x(0,t) = -qw_t(0,t) + pw_t^3(0,t), & t \geq 0, \\ w(1,t) = u(t), & t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in [0,1], \\ y(t) = w_t(0,t), & t \geq 0, \end{cases} \tag{1.1}$$

where p and q are given constants, w_0 and w_1 are initial condition in $t = 0$, u is boundary input at $x = 1$, and y is the boundary output measurement at $x = 0$. At $x = 0$ in system (1.1), the nonlinear boundary condition is known as the van der Pol type nonlinear boundary which can make different kind of dynamic behaviours, so, square wave, chaotic acoustic vibration, period-doubling bifurcation. If $p = 0$ and $q > 0$, then the uncontrolled system (1.1) is an unstable system and its stabilization problem has been studied in literature ([2]). Recently, the stabilization of (1.1) for $p,\ q > 0$ have been studied in [1].

2. Observer design

The aim of this section is to design an observer for wave equation (1.1). Thanks to output control plant (1.1), the following observer is proposed.

$$\begin{cases} \hat{w}_{xx}(x,t) = \hat{w}_{tt}(x,t), & x \in (0,1),\ t > 0, \\ \hat{w}_x(0,t) = -qw_t(0,t) + pw_t^3(0,t) + \alpha(w_t(0,t) - \hat{w}_t(0,t)), & t \geq 0, \\ \hat{w}(1,t) = u(t), & t \geq 0, \end{cases} \tag{2.1}$$

here α is the design gain. With the observer (2.1), the error $e(x,t) = w(x,t) - \hat{w}(x,t)$ governed by

$$\begin{cases} e_{xx}(x,t) = e_{tt}(x,t), & x \in (0,1),\ t > 0, \\ e_x(0,t) = \alpha e_t(0,t), & t \geq 0, \\ e(1,t) = 0, & t \geq 0. \end{cases} \tag{2.2}$$

The system (2.2), for $\alpha > 0$ is exponential stable ([2]).

3. Output feedback control design

The purpose of this section is to design an observer-based output feedback controller and to prove its exponential stability. To that end, the following transformation is introduced.

$$\tilde{w}(x,t) = [(I + P)\hat{w}](x,t) = \hat{w}(x,t) + Z(x,t),\ x \in [0,1],\ t \geq 0, \tag{3.1}$$

where $Z = P\hat{w}$ holds for following system

$$\begin{cases} Z_x(x,t) + Z_t(x,t) = 0 & x \in (0,1),\ t > 0, \\ Z(0,t) = -\beta\hat{w}(0,t) & t \geq 0, \\ Z(x,0) = Z_0(x) & x \in [0,1]. \end{cases} \tag{3.2}$$

So, according to (3.2), we have

$$Z(x,t)=\begin{cases}\beta\hat{w}(0,t-x), & t\geq x,\\ Z_0(x-t), & x>t.\end{cases}\tag{3.3}$$

Combining (2.1), (3.1) and (3.2), the following system is obtained

$$\begin{cases}\tilde{w}_{xx}(x,t)=\tilde{w}_{tt}(x,t),\ x\in(0,1),\ t>0,\\ \tilde{w}_x(0,t)=\dfrac{\beta-q}{1-\beta}\tilde{w}_t(0,t)+\dfrac{p}{(1-\beta)^3}\tilde{w}_t^3(0,t)+(\alpha-q)e_t(0,t)+pe_t^3(0,t)\\ \qquad\qquad+\dfrac{3}{(1-\beta)^2}\tilde{w}_t^2(0,t)e_t(0,t)+\dfrac{3}{1-\beta}\tilde{w}_t(0,t)e_t^2(0,t),\ t\geq 0,\\ \tilde{w}(1,t)=0,\ t\geq 0,\\ e_{xx}(x,t)=e_{tt}(x,t),\ x\in(0,1),\ t>0,\\ e_x(0,t)=\alpha e_t(0,t),\ t\geq 0,\\ e(1,t)=0,\ t\geq 0.\end{cases}\tag{3.4}$$

Lemma 3.1. *Let $\mathcal{H}=\{(f,g)\in H^1(0,1)\times L^2(0,1)\mid f(1)=0\}$, $\dfrac{\beta-q}{1-\beta}>0$, $\dfrac{p}{(1-\beta)^3}>0$, $\dfrac{\alpha(\beta-q)}{1-\beta}-\dfrac{(\alpha-q)^2}{2}>0$, $e_t(0,t)$, and $\hat{w}_t(0,t)$ be sufficiently small. Then, for any initial state $(\tilde{w}_0,\tilde{w}_1,e_0,e_1)\in\mathcal{H}^2$, system (3.4) accepts a unique solution $(\tilde{w}(.,t),\tilde{w}_t(.,t),e(.,t),e_t(.,t))\in C(0,\infty;\mathcal{H}^2)$ which satisfies*

$$\|\,(\tilde{w},\tilde{w}_t,e,e_t)\,\|_{\mathcal{H}^2}\leq Le^{-\omega t},\tag{3.5}$$

where L and ω are two positive constants.

Proof. These results can be concluded from [1, Lemma 2]. □

Using $u(t)=-Z(1,t)$, we obtain the following closed-loop of systems (1.1)

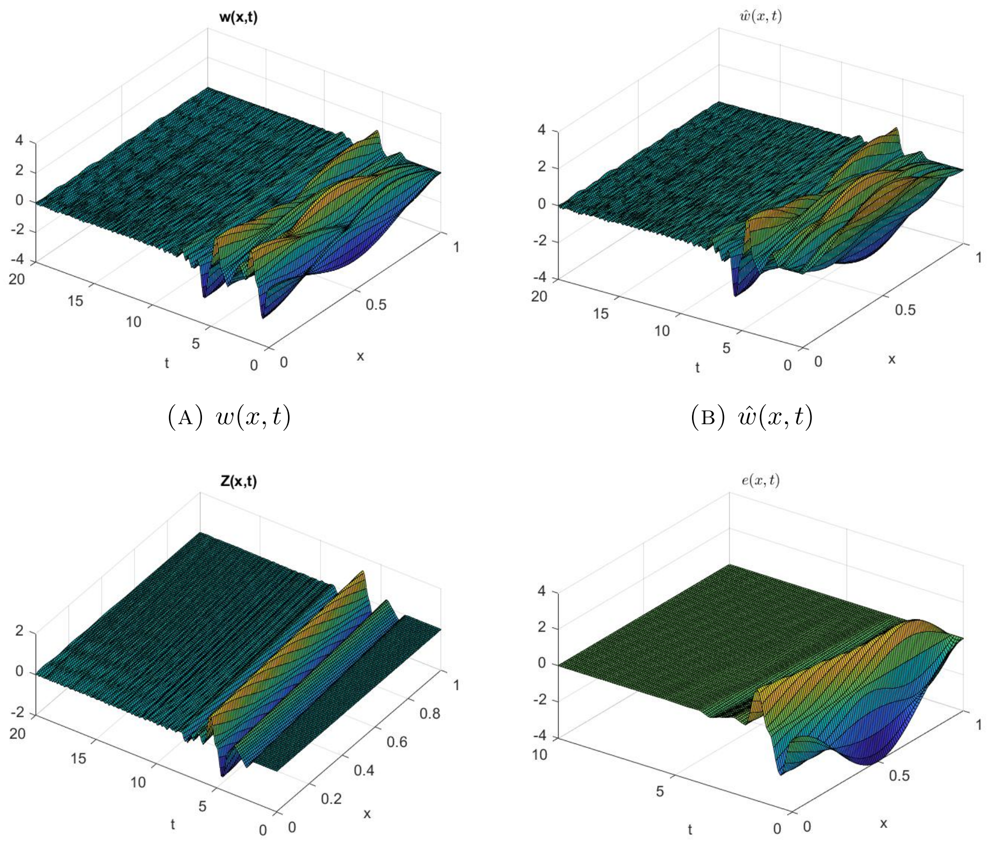
$$\begin{cases}w_{xx}(x,t)=w_{tt}(x,t), & x\in(0,1),\ t>0,\\ w_x(0,t)=-qw_t(0,t)+pw_t^3(0,t), & t\geq 0,\\ w(1,t)=-Z(1,t), & t\geq 1,\\ \hat{w}_{xx}(x,t)=\hat{w}_{tt}(x,t), & x\in(0,1),\ t>0,\\ \hat{w}_x(0,t)=-qw_t(0,t)+pw_t^3(0,t)+\alpha(w_t(0,t)-\hat{w}_t(0,t)), & t\geq 0,\\ \hat{w}(1,t)=-Z(1,t), & t\geq 1,\\ Z_t(x,t)+Z_x(x,t)=0,\quad Z(0,t)=-\beta\hat{w}(0,t), & x\in(0,1),\ t>0.\end{cases}\tag{3.6}$$

Theorem 3.2. *For any initial state $(w_0,w_1,\hat{w}_0,\hat{w}_1,Z)\in\left(H^1\times L^2\right)^2\times H^1$, system (3.6) accepts a unique solution and there are positive constants M and ϖ such that*

$$\|\,(w(.,t),w_t(.,t),\hat{w}(.,t),\hat{w}_t(.,t),Z(.,t))\,\|_{\mathcal{H}^2\times L^2}\leq Me^{-\varpi t}.\tag{3.7}$$

Proof. The proof immediately follows from (3.1), (3.3) and Lemma 3.1. □

To show the effectiveness of the proposed observer and stability of the closed-loop system, simulation of the error system (2.2), and output feedback system (3.6) are shown in Figure 1. The numerical results are implemented using Matlab R2017b. And we set, $\alpha = 0.83$, $\beta = 0.625$, $dx = 0.01$, $dt = 0.001$, $w(x,0) = \cos(2\pi x) - 1$, $\hat{w}(x,0) = -\cos(2\pi x) + 1$ and $e(x,0) = 2\cos(2\pi x) - 2$. Figure 1 shows that the effectiveness of the proposed observer and output feedback controller.



(A) $w(x,t)$ (B) $\hat{w}(x,t)$

(C) $Z(x,t)$ (D) $e(x,t)$

FIGURE 1. Simulation for error system (2.2) and output feedback systems (3.6)

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