NUMERICAL OPTIMAL CONTROL OF THE WAVE EQUATION

HASSAN ZAREI [1*], ALI ZAFARI [1]

[1] *Department of Mathematics, Payame Noor University,*

P.O. Box 19395-4697, Tehran, Iran.

zarei2003@gmail.com; zafari.math.pu@gmail.com

ABSTRACT. This paper, presents a spectral method approximating the boundary optimal control problem of the well-known wave equation with a linear optimal control problem. The method is based upon constructing the *M*th degree interpolation polynomials, using Chebyshevs nodes, to approximate the wave amplitude. By using the Pontryagin's maximum principle, necessary optimality conditions for resulting optimal control problem are derived and the optimal controls in piecewise constant form are obtained applying the control parameterization enhancing technique (CPET). Efficiency of the proposed method is confirmed by a numerical example.

1. INTRODUCTION

Consider the problem of minimizing the functional $J(\tau,\upsilon_1,\upsilon_2)=\int_0^\tau |\upsilon_1(t)|^p+|\upsilon_2(t)|^p\,dt$ subject to the one-dimensional wave equation $u_{tt}(z,t)=\alpha^2 u_{zz}(z,t),\ (z,t)\in(0,\ell)\times(0,\tau)$ with the initial conditions, $u(z,0)=u_1(z),\ \ u_t(z,0)=u_2(z),\ \ z\in(0,\ell)$ and the boundary conditions, $u(0,t)=\upsilon_2(t),\ \ u(\ell,t)=\upsilon_1(t),\ \ t\in(0,\tau)$ and the end conditions, $u(z,\tau)=s_1(z),\ \ u_t(z,\tau)=s_2(z),\ \ z\in(0,\ell)$ where υ_1 and υ_2 are measurable control functions which are assumed to be constrained as $\kappa_j\le\upsilon_j(t)\le\sigma_j,\ \ j=1,2,\ \ t\in(0,\tau)$, for, $\kappa_j<0$ and $\sigma_j>0$. A full discretization method based on appropriate finite differences is used to solve a special case of this problem in [3], where the functions s_1 and s_2 in end conditions are zero, the final time τ is fix and there is no box constraints on control functions. While, the problem considered in this paper is more general than those considered in [3]. Optimal control problems for wave equations are summarized in [6]. The spectral methods, as an effective tool, have been used to compute control problems for lumped [2] and, in recent years, distributed parameter systems [1], [7]. In this paper, we use the spectral method to solve the introduced problem in this section. The proposed method is outlined in the next section. In section 3, our problem is approximated with the optimal control problems and their

2010 *Mathematics Subject Classification.* 93C20, 35L05.

Key words and phrases. Chebyshev, control parameterization, optimal control, spectral method, wave equation.

[*] Speaker

optimality conditions are obtained, when p=1,2. Section 4 includes the numerical results obtained by the control parameterization enhancing technique.

2. THE PROPOSED METHOD

Let $T_M(x),\ x\in[-1,1]$ denote the Chebyshev polynomial of degree M, then the collocation points $x_j=\cos\left(\frac{\pi j}{M}\right),\ j=0,1,2,\ldots,M$, are the zeros of $(1-x^2)\dot{T}_M(x),\ x\in[-1,1]$. The Mth degree interpolation polynomials to $u(x,t)$ is given by

$$u^M(x,t)=\sum_{j=0}^{M}a_j(t)\,\varphi_j(x)\,, \tag{1}$$

where φ_j s are the Lagrange polynomials that $\varphi_j(x_k)=\delta_{kj}$ [2] . The relationship between $u^M(x,t)$ and $u^M_{xx}(x,t)$ at the Chebyshev nodes $x_k,\ k=0,1,2,...,M$ is given as $u^M_{xx}(x_k,t)=\sum_{j=0}^{M}d^{(2)}_{kj}a_j(t)$, where $D^{(2)}=\left(d^{(2)}_{kj}\right)$ is the second order Chebyshev derivative matrix [10]. In the next section we approximate the constraints of the above mentioned problem with a linear control system applying the proposed method.

3. OPTMAL CONTROL FORMULATION

In order to use the Chebyshev nodes we introduce the transformation $z=\frac{\ell}{2}(1+x)$. In this way the wave equation introduced in section 1 convert to $u_{tt}(x,t)=\beta^2u_{xx}(x,t),\ (x,t)\in(-1,1)\times(0,\tau)$ with the initial conditions, $u(x,0)=u_1(x),\quad u_t(x,0)=u_2(x),\quad x\in(-1,1),$ and the boundary conditions, $u(-1,t)=\upsilon_2(t),\quad u(1,t)=\upsilon_1(t),\quad t\in(0,\tau),$ and the end conditions, $u(x,\tau)=s_1(x),\quad u_t(x,\tau)=s_2(x),\quad x\in(-1,1),$ where $\beta=\frac{2}{\ell}\alpha$. Considering the function u in the form of (1) we get a linear second order controlled system as

$$\ddot{a}(t)=Ca(t)+Bv(t)\,,\ a(0)=a_i\ ,\dot{a}(0)=a_i'\,,\ a(\tau)=a_f\ ,\dot{a}(\tau)=a_f'\,, \tag{2}$$

where,

$$C=\beta^2\begin{bmatrix} d^{(2)}_{11} & \cdots & d^{(2)}_{1M-1} \\ \vdots & \ddots & \vdots \\ d^{(2)}_{M-11} & \cdots & d^{(2)}_{M-1M-1}\end{bmatrix},\quad B=\beta^2\begin{bmatrix} d^{(2)}_{10} & d^{(2)}_{1M} \\ \vdots & \vdots \\ d^{(2)}_{M-10} & d^{(2)}_{M-1M}\end{bmatrix},$$ $a(t)=(a_1(t),\ldots,a_{M-1}(t))^T$, $V(t)=(v_1(t)\ ,v_2(t))^T$,

$a_i=(u_1(x_1),\ldots,\ u_1(x_{M-1}))^T$, $a_i'=(u_2(x_1),\ldots,\ u_2(x_{M-1}))^T$, $a_f=(s_1(x_1),\ldots,\ s_1(x_{M-1}))^T$ and $a_f'=(s_2(x_1),\ldots,\ s_2(x_{M-1}))^T$. Introducing a new variable $b(t)=\dot{a}(t)$, the control system (2) can be written as a following first order linear controlled system

$$\dot{Y}(t)=EY(t)+FV(t),\quad Y(0)=\begin{bmatrix}a_i'\\ a_i\end{bmatrix},\ Y(\tau)=\begin{bmatrix}a_f'\\ a_f\end{bmatrix}, \tag{3}$$

where, $Y(t)=\begin{bmatrix}b(t)\\ a(t)\end{bmatrix}$, $E=\begin{bmatrix}0 & C\\ I & 0\end{bmatrix}$ and $F=\begin{bmatrix}B\\ 0\end{bmatrix}$.

3.1 Minimum energy problem. As a minimum energy problem (MEP), we first consider the objective functional $J(\tau,\upsilon_1,\upsilon_2)=\int_0^\tau \upsilon_1^2(t)+\upsilon_2^2(t)\,dt$. According to Pontryagin's Maximum Principle [4], the necessary optimality conditions for optimal controls $V^*(t)=(v_1^*(t),v_2^*(t))^T$ and the optimal time τ^* minimizing the functional J subject to constraints (3) is

$$v_j^*(t)=\max\{\kappa_j,\min\{\sigma_j,-\tfrac{1}{2}\lambda_1^*(t)B_j\}\},\ H(a^*(\tau^*),b(\tau^*),\upsilon_1^*(\tau^*),\upsilon_2^*(\tau),\lambda_1^*(\tau^*),\lambda_2^*(\tau^*))=0, \quad (4)$$

where B_j, $j=1,2$ is the jth column of B, H is the Hamiltonian defined as $H(a,b,\upsilon_1,\upsilon_2,\lambda_1,\lambda_2)=\upsilon_1^2+\upsilon_2^2+\lambda_1(Ca+BV)+\lambda_2 b$, λ_1^* and λ_2^* are the costate variables which satisfy

$$\dot{\lambda}_1=-\frac{\partial H}{\partial b}=-\lambda_2,\ \dot{\lambda}_2=-\frac{\partial H}{\partial a}=-\lambda_1 C\ , \quad (5)$$

and $b^*(t)$ and $a^*(t)$ are solution of (3) corresponding to $V^*(t)$.

3.2 Minimum fuel problem. Now, we consider the minimum fuel problem (MFP) by setting $J(\tau,\upsilon_1,\upsilon_2)=\int_0^\tau |\upsilon_1(t)|+|\upsilon_2(t)|\,dt$. The Pontryagin's Maximum Principle gives the following necessary condition for optimal controls and time

$$\upsilon_j^*(t)=\begin{cases}\sigma_j & \lambda_1^*(t)B_j\le -1,\\ 0 & -1<\lambda_1^*(t)B_j<1,\\ \kappa_j & \lambda_1^*(t)B_j\ge 1,\end{cases}\ H(a^*(\tau^*),b(\tau^*),\upsilon_1^*(\tau^*),\upsilon_2^*(\tau),\lambda_1^*(\tau^*),\lambda_2^*(\tau^*))=0, \quad (6)$$

where, $H(a,b,\upsilon_1,\upsilon_2,\lambda_1,\lambda_2)=|\upsilon_1|+|\upsilon_2|+\lambda_1(Ca+BV)+\lambda_2 b$, λ_1^* and λ_2^* are the costate variables satisfying at (5) and $b^*(t)$ and $a^*(t)$ are solution of (3) corresponding to $V^*(t)$. It can be proved that if the matrices $[B_j\,|CB_j\,|\cdots|C^{M-1}B_j]$, j=1, 2 are nonsingular, then there is no singular intervals and the optimal controls are completely determined by (6) (See [4] for more details). We point out that the optimality system for both the MEP and the MFP consists of the state system (3) with the boundary conditions, the costate system (5), together with the expressions (4) and (6) for the control functions and optimal time. Due to difficulties in solving the optimality systems for MEP and MFP, in the next section we, directly use the control parameterization enhancing technique (CPET) introduced in [5] to optimize the functional $J(\tau,\upsilon_1,\upsilon_2)$ subject to the linear control system (3).

4. NUMERICAL RESULTS

As a numerical example, we consider a problem with $\ell=100$, $\alpha=1$, $u_1(z)=\frac{4}{\ell^2}\left(z-\frac{\ell}{2}\right)^2$, $u_2(z)=0$, $s_1(z)=\sin\left(\frac{2\pi z}{\ell}\right)$, $s_2(z)=0$, κ_j=-1, σ_j =1, j=1,2. In our implementation we set M=10. Figures 1 and 2, respectively, show the optimal boundaries $\upsilon_j^*, j=1,2$ and the corresponding optimal state $u^M(x,t)$ for MEP obtained by CPET. The optimal value for objective functional (6) obtained by CPET is $J(\tau^*,\upsilon_1^*,\upsilon_2^*)=$ 23.3801 which is corresponding to

$\tau^* = 147.1297$. The optimal controls obtained by CPET for MFP and the corresponding optimal state $u^M(x,t)$ are depicted in Figures 3 and 4, respectively. Moreover, the corresponding objective functional value is $J(\tau^*, \upsilon_1^*, \upsilon_2^*) = 58.1754$ which is corresponding to $\tau^* = 50.8477$.

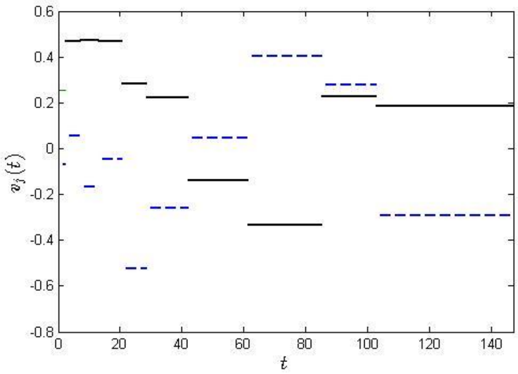

Figure 1. Control functions v_1(- - -) and v_2(–) for MEP obtained by CPET1

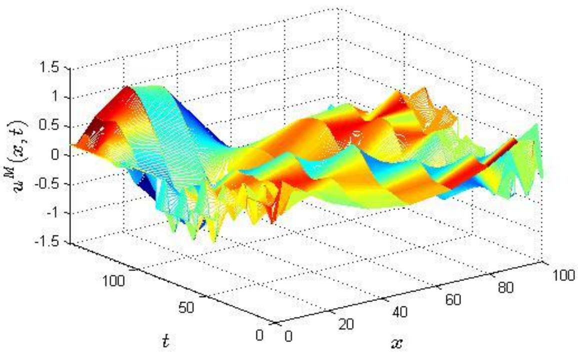

Figure 2. u^M(x,t) for MEP obtained by CPET1

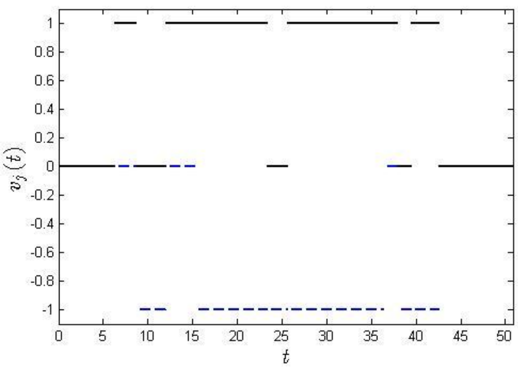

Figure 3. Control functions v_1(---) and v_2(–) for MFP obtained by CPET2

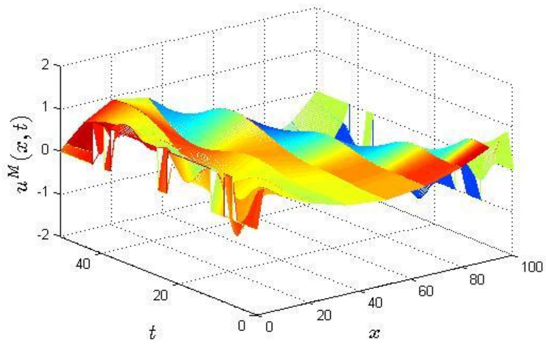

Figure 4. u^M(x,t) for MFP obtained by CPET2.

REFERENCES

1. Y. Chen, F. Huang, N. Yi, W. Liu, *A Legendre-Galerkin spectral method for optimal control problems governed by stokes equations*, Siam J. Numer. Anal., **49 (2011)** 1625–1648.

2. G .N. Elnagar, M. Razzaghi, *A chebyshev spectral method for the solution of nonlinear optimal control problems*, Appl. Math. Modelling, **21** (1997) 255-260.

3. M. Gerdts, G. Greif, H. J. Pesch, *Numerical optimal control of the wave equation: optimal boundary control of a string to rest in finite time*, Math. Comput. Simuat., **79** (2008) 1020-1032.

4. D. E. Kirk, *Optimal Control Theory: An Introduction*, Prentice Hall, New York, NY, USA, 1970.

5. H. W. J. Lee, K. L. Teo, L. S. Jennings, V. Rehbock, *Control parametrization enhancing technique for optimal discrete-valued problems*, Automatica, **35** (1999), 1401-1407.

6. L. A. Manita, *Optimal control problems for wave equations*, J. Math. Sci., **177** (2008) 1324–11331.

7. H. Zarei, *A chebyshev spectral method for optimal control of the heat equation*, 2nd National Conference on Mathematics and its Applications, (2015) 1-4.