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A NOVEL ALGORITHM FOR SOLVING FUZZY MAXIMAL FLOW PROBLEMS

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Abstract. The aim of this paper is to develop a new method named as the *fuzzy pre flow-push algorithm* to solve the maximal flow problem under fuzzy environment. To this end, by considering a ranking function and any type of fuzzy number, we improved the pre flow-push algorithm. Moreover, a numerical example is given to illustrate that the proposed method offers a powerful tool to deal with such problems.

1. Introduction

In optimization theory, maximum flow problems involve finding a maximum flow among feasible flows in the network. In reality, some coefficients of a system are uncertain and only some imprecise information about the actual value of the parameters are available. So, the traditional methods cannot be used to preserve such properties of the maximum flow problem under uncertain environment. In order to deal with uncertainty, various non-probabilistic and probabilistic approaches have been developed, which differ by their advantages and disadvantages [5]. Here, we seek to improve the pre-flow push algorithm to find the maximum flow of a network, where the flow capacity of each arcs is an arbitrary fuzzy numbers.

We will denote the set of all fuzzy numbers by $\mathcal{F}(\mathbb{R})$, which is the class of fuzzy subsets of the real axis $\tilde{U} : \mathbb{R} \to [0, 1]$ satisfying normal, fuzzy convex, upper semicontinuous and compactly. The notation $[\tilde{A}]_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$ denotes explicitly the α-level set of fuzzy number $\tilde{A}$, where $\underline{a}_\alpha$ and $\overline{a}_\alpha$ are the lower and upper bounds, respectively. A special type of fuzzy number is the ***Gaussian fuzzy number*** $\tilde{A}$, which it's α-level sets are obtained as

$$[\tilde{A}]_\alpha = [x_1 - \sigma_l\sqrt{-2ln(\alpha)},\ x_1 + \sigma_r\sqrt{-2ln(\alpha)}].$$

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Moreover, a fuzzy number $\tilde{A}$ is called non-negative, denoted by $\tilde{A} \succeq 0$, if its membership function $\tilde{A}(x)$ satisfies $\tilde{A}(x) = 0,\ \forall x \leq 0$.

For two arbitrary fuzzy numbers $\tilde{A},\ \tilde{B}$ and a real number λ, the fuzzy arithmetic $\tilde{A} \circledast \tilde{B}$ and $\lambda.\tilde{A}$ with $* \in \{+, -, \times, \div\}$ are defined in terms of their $\alpha-$level sets as:

$$[\tilde{A} \circledast \tilde{B}]_\alpha = \{a * b \mid a \in [\tilde{A}]_\alpha,\ b \in [\tilde{B}]_\alpha\},\ [\lambda.\tilde{A}]_\alpha = \{\lambda a \mid a \in [\tilde{A}]_\alpha\},$$

where $[\tilde{A}]_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha], [\tilde{B}]_\alpha = [b^-_\alpha, b^+_\alpha]$ and $0 \notin [\tilde{B}]_0$ in the division $\tilde{A} \div \tilde{B}$.

The above difference (Minkowski difference) has a drawback that $\tilde{A} \ominus \tilde{A} \neq 0$. To partially overcome this deficiency, two other differences were introduced named as "H-difference" and "gH-difference", but these do not always exist. To solve this shortcoming, generalized difference between fuzzy numbers was defined by Stefanini [6], which was modified by Gomes [4] and the following result was obtained

$$[\tilde{A} \ominus_g \tilde{B}]_\alpha = \left[\inf_{\beta \geq \alpha} \min\{\underline{a}_\beta - \underline{b}_\beta, \overline{a}_\beta - \overline{b}_\beta\}, \sup_{\beta \geq \alpha} \max\{\underline{a}_\beta - \underline{b}_\beta, \overline{a}_\beta - \overline{b}_\beta\}\right].$$

Briefly, the g-difference is a generalization of gH-difference and gH-difference generalizes the H-difference. Therefore, if the H-difference (gH-difference) of two fuzzy numbers exists, the two other difference (g-difference) also exist and have the same value [2].

There are various techniques to rank fuzzy numbers, for example, Ezzati et.al [3] were proposed the rank function $\mathcal{R} : \mathcal{F}(\mathbb{R}) \times \{0, 1\} \to \mathbb{R}$ for an arbitrary $\tilde{A} \in \mathcal{F}(\mathbb{R})$ with $[\tilde{A}]_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$ as

$$\mathcal{R}(\tilde{A}, \gamma) = Mag(\tilde{A}) + \gamma Momag(\tilde{A}) \tag{1.1}$$

where

$$Mag(\tilde{A}) = \frac{1}{2}\int_0^1 (\underline{a}_\alpha + \overline{a}_\alpha + \overline{a}_1 + \underline{a}_1)\alpha d\alpha, \quad Momag(\tilde{A}) = \frac{1}{2}\int_0^1 (\underline{a}_\alpha - \overline{a}_\alpha + \overline{a}_1 - \underline{a}_1) d\alpha$$

and

$$\gamma = \begin{cases} 0, & Mag(\tilde{A}) \neq Mag(\tilde{B}), \\ 1, & Mag(\tilde{A}) = Mag(\tilde{B}) \text{ and } z_0 \geq 0, \\ -1, & Mag(\tilde{A}) = Mag(\tilde{B}) \text{ and } z_0 < 0, \end{cases}$$

which $z_0 = \frac{\overline{a}_1 + \underline{a}_1}{2}$ or $z_0 = \frac{\overline{b}_1 + \underline{b}_1}{2}$. Therefore, to compare two fuzzy numbers $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$, we have

$$\tilde{A} \succ (\prec)\tilde{B} \iff \mathcal{R}(\tilde{A}, \gamma) > (<)\mathcal{R}(\tilde{B}, \gamma).$$

2. Main results

Here, an algorithm based on the well-known pre flow-push algorithm for calculating the fuzzy maximal flow in a fuzzy network is proposed and is named as ***fuzzy pre flow-push algorithm.*** Whereas, the algorithm decomposes the operation of sending δ units of flow along a path into a set of basic operations of sending δ unit of flow along each arc of the path, which identified as a ***push***. Consider a directed graph $G = (V, E)$ with single-source 1 and single-sink n, where $V = \{1, 2, ..., n\}$ is a finite set of n nodes, and $E = \{(i, j) \mid i, j \in V\}$ is the set of m arcs. Associating to each arc (i, j), two fuzzy value

$\tilde{u}_{ij}$, $\tilde{x}_{ij}$ denote the amount of fuzzy capacity and fuzzy flow on the arc (i,j), respectively. If $\tilde{f}$ represent the amount of fuzzy maximal flow from source node 1 to sink node n, the ***fuzzy maximal flow problem*** can be modeled as:

$$\begin{aligned}
&\max\ \tilde{f}\\
&S.t.\ \sum_j \tilde{x}_{1j} \ominus_g \sum_k \tilde{x}_{k1} = \tilde{f},\\
&\qquad \sum_j \tilde{x}_{ij} \ominus_g \sum_k \tilde{x}_{ki} = \tilde{0},\ \forall i \in V \setminus \{1,n\},\\
&\qquad \sum_k \tilde{x}_{kn} \ominus_g \sum_j \tilde{x}_{nj} = \tilde{f},\\
&\qquad 0 \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij},
\end{aligned}$$

For a given fuzzy preflow $\tilde{x}$, the fuzzy excess of node $i \in V$ is a fuzzy number obtained by

$$\tilde{e}(i) = \sum_j \tilde{x}_{ji} \ominus_g \sum_k \tilde{x}_{ik}.$$

Obviously, in a fuzzy preflow, $\tilde{e}(i) \succeq 0$ for each node $i \in V \setminus \{1\}$ and $\tilde{e}(1) \preceq 0$. A node with positive excess is called an ***active node*** and we contracted out that the sink and source nodes are never active. The fuzzy pre flow-push algorithm contains two basic operation i) finding an active node and sending flow along an admissible arc ii) updating distance label by using a labeling algorithm [1], whereas for a fuzzy flow $\tilde{x}$,

1. the fuzzy residual capacity $\tilde{r}_{ij}$ for any arc $(i,j) \in E$, is a fuzzy number which is defied as
$$\tilde{r}_{ij} = (\tilde{u}_{ij} \ominus_g \tilde{x}_{ij}) \oplus \tilde{x}_{ji},$$
and a ***fuzzy residual network*** $G(x)$ is a directed graph $G(V,E)$ with the residual capacity on its arc,
2. an arc (i,j) in the fuzzy residual network is called ***admissible arc*** if its labels satisfies $d(i) = d(j) + 1$.

Precisely, the process of the fuzzy pre flow-push algorithm can be presented as

Begin
  $x_{ij} = \tilde{0}$;
  For each node i, compute the distance label $d(i)$;
  For each node j adjacent to source node 1, put $\tilde{x}_{1j} = \tilde{u}_{1j}$ and $d(1) = n$;
  While "the network contain an active node" do
    Begin
      Select an active node i;
      If "the network contains an admissible arc (i,j)" then
        Begin
          Push $\tilde{\delta} = \min\{\tilde{e}(i), \tilde{r}_{ij}\}$ units of flow from node i to node j;
          Updating the excess value of node i and j as $\tilde{e}(i) \ominus_g \tilde{\delta}$ and $\tilde{e}(j) \oplus \tilde{\delta}$, respectively;
          Updating the residual capacity of arc (i,j) and (j,i) as $\tilde{r}_{ij} \ominus_g \tilde{\delta}$ and $\tilde{r}_{ji} \oplus \tilde{\delta}$, respectively;
        End
      Else
          Replace the label of node i by $\min\{d(j)+1 | (i,j) \in A(i),\ \tilde{r}_{ij} \succ \tilde{0}\}$;
    End
End

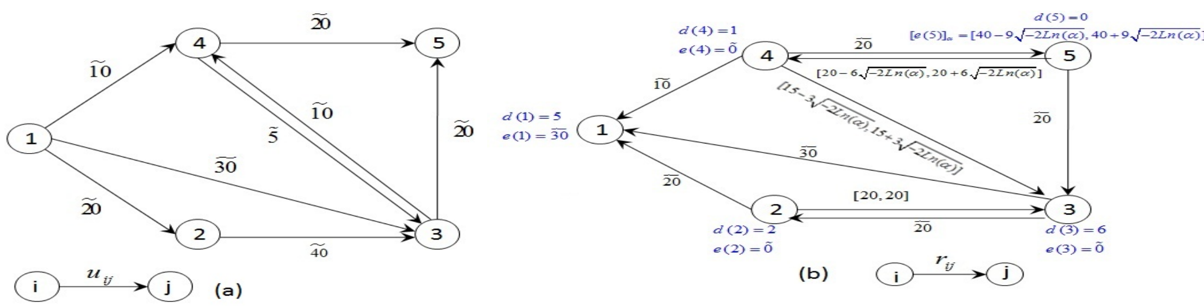

FIGURE 1. (a) Initial fuzzy network (b) Final fuzzy network.

where $A(i)$ is the ***adjacency list*** that includes the arcs emanating from node i.

Example 2.1. Consider the following network with Gaussian fuzzy numbers as its arc capacity and the source node 1 and sink node 5 (Figure 1a), where

$$[\tilde{5}]_\alpha = [5 - 1.5\sqrt{-2ln(\alpha)},\ 5 + 1.5\sqrt{-2ln(\alpha)}],\ [\tilde{10}]_\alpha = [10 - 3\sqrt{-2ln(\alpha)},\ 10 + 3\sqrt{-2ln(\alpha)}],$$
$$[\tilde{20}]_\alpha = [20 - 3\sqrt{-2ln(\alpha)},\ 20 + 3\sqrt{-2ln(\alpha)}],\ [\tilde{30}]_\alpha = [30 - 4.5\sqrt{-2ln(\alpha)},\ 30 + 4.5\sqrt{-2ln(\alpha)}],$$
$$[\tilde{40}]_\alpha = [40 - 3\sqrt{-2ln(\alpha)},\ 40 + 3\sqrt{-2ln(\alpha)}].$$

By applying the **Fuzzy pre flow-push algorithm**, the fuzzy maximal flow is obtained as $[\tilde{40}]_\alpha = [40 - 9\sqrt{-2ln(\alpha)}, 40 + 9\sqrt{-2ln(\alpha)}]$. The iterations of the algorithm are summarized in the Table 1, which is shown in Figure 1b the active node and admissible arc and the α-level set of push value in each iteration.

TABLE 1. Iterations of Example 2.1.

Iterations	Active node	Admissible arc	α-cut of the push value
1	3	$(3,5)$	$[20 - 3\sqrt{-2ln(\alpha)},\ 20 + 3\sqrt{-2ln(\alpha)}]$
2	3	$(3,4)$	$[10 - 3\sqrt{-2ln(\alpha)},\ 10 + 3\sqrt{-2ln(\alpha)}]$
3	4	$(4,5)$	$[20 - 6\sqrt{-2ln(\alpha)},\ 20 + 6\sqrt{-2ln(\alpha)}]$
4	2	$(2,3)$	$[20 - 3\sqrt{-2ln(\alpha)},\ 20 + 3\sqrt{-2ln(\alpha)}]$
5	3	$(3,1)$	$[30 - 4.5\sqrt{-2ln(\alpha)},\ 30 + 4.5\sqrt{-2ln(\alpha)}]$

REFERENCES

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows: Theory, algorithms, and applications*, 1993.
[2] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, 2013.
[3] R. Ezzati, S. Khezerloo, and S. Ziari, *Application of parametric form for ranking of fuzzy numbers*, Iranian Journal of Fuzzy Systems, 12(1) (2015), 59-74.
[4] L. T. Gomes and L. C. Barros, *A note on the generalized difference and the generalized differentiability*, Fuzzy Sets and Systems, 2015.
[5] S. Majumder and A. Pal, *A new algorithm for solving maximal flow problem in an intuitionistic fuzzy network*, Software Engineering and Technology, 6(1) (2014), 25-34.
[6] L. Stefanini. *A generalization of hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy sets and systems, 161(11) (2010), 1564-1584.