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MINIMUM DISTORTION EMBEDDING OF SHAPS IN 3D-EUCLIDEAN SPACE

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Abstract. In order to say whether two shapes are similar, we compare them as metric spaces. From the point of view of metric geometry, two metric spaces are equivalent if their corresponding metric structures are symmetric. Based on types of distances, symmetry can be classified into extrinsic and intrinsic. While extrinsic symmetry detection finds rigid transformations that are Euclidean distance preserving, intrinsic symmetry detection of 3D shapes, using geodesic distances has received considerable attentions, as a problem in recent years. Among the several challenging soloutions , there is an approach to find the minimum distortion embedding of the shape into a finite dimentional Euclidean space and the analysis of the intrinsic symmetry group of the shape can be reduced to the analysis of some extrinsic symmetry group. In this paper, we review this problem and its soloution, by proposing a nonlinear optimization problem as an efficient approach in this way.

1. Introduction

Symmetries are universal phenomena, in both natural and manmade shapes, which reflect high-level information about shape structure. Many applications in geometric modeling and processing utilize the symmetry information. The characterization and detection of symmetries of 3D shapes, thus receives significant attention in computational geometry and computer graphics. Generally speaking, approaches for shape similarity through the metric geometry framework model the shapes as metric spaces equipped with some metric and so, the symmetry of a shape can be regarded as a distance preserving self-homeomorphism. Based on types of distances, symmetry can be classified into extrinsic and intrinsic. While extrinsic symmetry detection finds rigid transformations that are Euclidean distance preserving, intrinsic symmetry detection uses geodesic distances and looks

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for isometric deformations. The simplest choice is the Euclidean metric arising from the space $\mathbb{R}^3$ in which the shape X is embedded, which is invariant to Euclidean isometries (the elements of the isometry group $\mathbf{Iso}(\mathbb{R}^\mathbf{3}, \mathbf{d}\mathbb{R}^\mathbf{3})$ are rigid motions including rotations, translations and reflections). The metric space $(X, d\mathbb{R}^3)$ is a subset of the metric space $(\mathbb{R}^3, d\mathbb{R}^3)$. Given two shapes $(X, d\mathbb{R}^3)$ and $(Y, d\mathbb{R}^3)$ and regarding them as subsets of $(\mathbb{R}^3, d\mathbb{R}^3)$, their similarity can be quantified using the Hausdorff distance

$$d_H(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d(x, y), \sup_{y\in Y} \inf_{x\in X} d(x, y)\}$$

which expresses the similarity between two subsets of a metric space with metric d. Since the shapes are defined up to Euclidean isometry, one minimizes d_H over all the possible rigid motions,

$$\min_{i\in Iso(\mathbb{R}^3, d\mathbb{R}^3)} d_H(i(X), Y),$$

parametrized by a small number of degrees of freedom (three rotation angles and three translation coordinates). This optimization problem can be regarded as best possible rigid alignment of X and Y in $\mathbb{R}^3$, and is solved efficiently using iterative closest point (ICP) algorithms (provided that good initialization parameters are known). In a more general setting, we are given two shapes (X, d_X) and (Y, d_Y) with some generic metrics d_X, d_Y (e.g. the geodesic or diffusion metrics invariant to isometric deformations of the shapes) that do not arise from a common metric space. In this case, one can either try to compare the metric directly or alternatively, reduce the problem to the aforementioned setting. For this purpose, one tries to represent the metric d_X (respectively, d_Y) in some fixed metric space (Z, d_Z) by means of an isometric embedding $f : X \to Z$ (respectively, $g : Y \to Z$) satisfying $d_X = d_Z \circ (f \times f)$ (respectively, $d_Y = d_Z \circ (g \times g)$). The images $f(X)$ and $g(Y)$, referred to as canonical forms by Elad and Kimmel [EK03], can be compared as subsets of (Z, d_Z) using the Hausdorff distance under the isometries in (Z, d_Z),

$$\min_{i\in Iso(Z, d_Z)} d_H(i(f(X)), g(Y)).$$

The choice of the embedding space (Z, dZ) should be such that its isometries can be easily parametrized and searched over. In particular, when $Z = \mathbb{R}^3$, the comparison of canonical forms boils down to the rigid alignment problem. Unfortunately, isometric embeddings of general metrics into a Euclidean space typically do not exist. It is however possible to find the best possible approximate isometry, by minimizing some error criterion Bi

$$\min_{i\in Iso(Z, d_Z)} \parallel d_X - d_Z \circ (f \times f) \parallel . \tag{1.1}$$

Elad and Kimmel [**?**] used the L^2 (least-squares) error, finding the approximately isometric embedding by solving the multi-dimensional scaling (MDS) problem [**?**]. We review problem as an application of nonlinear optimization theory in computer graphic and other similar filds.

2. Sketch of the problem

Triangulation. Given surfaces in 3D, we would like to measure their isometric dissimilarity and thereby classify them. At the first step, we compute the geodesic distance matrix for each surface on triangulated domains. A triangulated surface is an approximate representation of a continuous one. The representation error introduced by triangulating a smooth surface is of an order of the length of the edges of the triangles. The smaller the triangles get, the more accurate the triangulation represents the surface. In order to compute the geodesic distances between pairs of points on the surface, we use a method, the fast marching on triangulated domains .The basic idea is an efficient numerical approach that solves an Eikonal equation on the triangulated surface. The solution is a surface distance function that is proven to converge to the viscosity smooth solution as the numerical grid (triangulation) is refined. Like the Dijkstra graph search method, the distance function is constructed by starting from a sources point and propagating outwards. The fast marching on triangulated domains method can compute the geodesic distance between one vertex and the rest of the $\widehat{n}$ surface vertices in $O(\widehat{n})$ operations. Repeating this computation for $n(n < \widehat{n})$ selected vertices, we can compute a geodesic distance matrix D in $O(n\widehat{n})$ operations.Eachij entry of D represents the square geodesic distance between vertex v_i and vertex v_j. That is

$$\delta_{ij} = d_S(v_i, v_j), \qquad D_{ij} = (\delta_{ij})^2.$$

where $d_S(v_i, v_j)$ is the geodesic distance (measured on the surface S) between the surface point indicated by the vertex v_i, and the surface point indicated by the vertex v_j.

Multi-Dimensional Scaling Method. Multi-Dimensional Scaling (MDS) is a family of methods that map measurements of similarity or dissimilarity among pairs of feature items into distances between feature points with given coordinates in a small-dimensional Euclidean space. The graphical display of the (di)similarity measurements provided by an MDS procedure enables us to view the data and explore its geometric structure. Most metrical MDS methods expect a set of n items and their pairwise (dis)similarities and the desired dimensionality, m, of the Euclidean embedding space.

Definition 2.1. [**?**] MDS algorithms map each item to a point $x_i = X_i$ in an $m-$dimensional Euclidean space R^m by minimization of the function $S(X)$ as follows:

$$S(X) = \frac{\sum_{i<j} \omega_{ij}(\delta_{ij} - d_{ij}(X))^2}{\sum_{i<j}(\delta_{ij})^2} \tag{2.1}$$

where δ_{ij} is the input dissimilarity measure between item i and j, $d_{ij}(X)$ is the Euclidean distance between these items in the $m-$dimensional Euclidean space, and ω_{ij} are some weighting coefficients. the function $S(X)$ is called the stress function.

The bending invariant representation is constructed by first measuring the intergeodesic distances between uniformly distributed points on the surface. Next, a type of multidimensional scaling (Least Squares MDS) technique is applied to extract coordinates in a finite dimensional Euclidean space in which geodesic distances are replaced by Euclidean ones.

Applying this transform to various surfaces with similar geodesic structures (first fundamental form) maps them into similar signature surfaces. We thereby translate the problem of matching nonrigid objects in various postures into a simpler problem of matching rigid objects as the final result. As mentioned in **??** and **??** , to detect the intrinsic symmetries, we must minimize the stress function. the following theorem provid this opportunity:

Theorem 2.2. *Bo, El Minimizing the stress function* **??** *is equivalent to minimizing the following functional:*

$$S(X) = \sum_{i<j} \omega_{ij}(\delta_{ij} - d_{ij})^2$$

where

$$d_{ij}(X) = (\sum_{a=1}^{n}(x_{ia} - x_{ja})^2)^{\frac{1}{2}}$$

and the following is the final coordinates:

$$X_i = n^{-1}B(Z)Z$$

where the matrix $B(Z)$ elements are

$$b_{ij}(Z) = \frac{-\delta ij}{d_{ij}(Z)}$$

for $i \neq j$ and otherwise is 0 and also

$$b_{ii}(Z) = \sum_{j=1,j}^{n} b_{ij}.$$

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