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FRACTIONAL LANGEVIN EQUATION WITH ANTI-PERIODIC BOUNDARY CONDITIONS, BY MEANS OF A COUPLED FIXED POINT THEOREM

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Abstract. In this paper, we study a nonlinear Langevin equation involving two fractional orders $\alpha \in (1, 2]$, $\beta \in (1, 2]$ with intial conditions. By means of an interesting coupled fixed point theorem, we establish sufficient conditions for the existence and uniquensess of solutions for some fractional equations with different boundary conditions

1. Introduction

In this paper, we study the existence and uniqueness of solutions for the following boundary value problem of Langevin equation with two different fractional orders:

$$\begin{cases} D^{\beta}(D^{\alpha}+\lambda)x(t) = f(t, x(t)), \quad 0 \le t \le 1, \quad 1 < \alpha \le 2, \quad 1 < \beta \le 2, \\ x(0) = 0, x'(0) = 1, x(1) = \int_0^{\eta} x(\tau)d\tau \text{ for some } 0 < \eta < 1, \mathcal{D}^{2\alpha}x(1) + \lambda \mathcal{D}^{\alpha}x(1) = 0, \end{cases} \tag{1.1}$$

where D^{α} is the Caputo fractional derivative of order α, $f : [0, 1]\times \to$ is a given continuous function and λ is a real number. Furthermore, $\mathcal{D}^{2\alpha}$ is the sequential fractional derivative

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$$\begin{cases} \mathcal{D}^{\alpha}u = D^{\alpha}u, \\ \mathcal{D}^{k\alpha}u = \mathcal{D}^{\alpha}\mathcal{D}^{(k-1)\alpha}u, k = 2, 3, \cdots . \end{cases}$$

1.1. **Preliminaries.**

Definition 1.1. The Riemann-Liouville fractional integral of order $\alpha \in^{+}$ for a continuous function $x : [0, \infty) \rightarrow$ is defined as

$$I^{\alpha}x(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad 0 \leq t \leq 1, \tag{1.2}$$

where $\Gamma(.)$ is the Gamma function, provided that the right-hand-side integral exists and be finite.

Definition 1.2. For a continuous function $x : [0, \infty) \rightarrow$, the Caputo derivative of fractional order $\alpha \in^{+}$ is defined as

$$D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} x^{(n)}(s)ds, \quad 0 \leq t \leq 1, \tag{1.3}$$

where $n - 1 < \alpha < n$, provided that the right-hand-side integral exists and be finite.

Lemma 1.3. *Let $\alpha, \beta \geq 0$. If x is continuous, then $I^{\alpha}I^{\beta}x = I^{\beta}I^{\alpha}x = I^{\alpha+\beta}x$.*

Lemma 1.4. *Let $\alpha \geq 0$. If x is continuous, then $D^{\alpha}I^{\alpha}x = x$.*

Lemma 1.5. *Let $n \in \mathbb{N}^{+}$, $\alpha \in (n-1, n)$. Then, the general solution of the fractional differential equation $D^{\alpha}x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in, i = 1, 2, \cdots, n$.

Theorem 1.6. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X. Assume thatthere exists a $k \in [0, 1]$ with*
$d(f(x, y), f(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$,
for each $x \succeq u, y \preceq v$. Suppose either f is continuous or X has the following property:
(i) if a non-decreasing sequece $\{x_n\}$,then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequece $\{y_n\}$,then $y \preceq y_n$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$, than f has a coupled fixed point $(x^*, y^*) \in X$.

2. Existence and uniqueness results

Lemma 2.1. *x is a solution of the problem (1.1) if and only if it is a solution of the nonlinear integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau,x(\tau))d\tau - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau \\ &+ t^\alpha\Big[\int_0^\eta x(\tau)d\tau - 1 + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1} f(\tau,x(\tau))d\tau \\ &+ \frac{\Gamma(1-\alpha)}{\Gamma(\alpha+2)} + \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta (1-\tau)^{\alpha-1}x(\tau)d\tau \\ &- \frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-\tau)^{\alpha+\beta-1} f(\tau,x(\tau))d\tau\Big] \\ &- \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\Big[\frac{\Gamma(2-\alpha)}{\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1} f(\tau,x(\tau))d\tau + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}\Big] + t. \end{aligned} \tag{2.1}$$

Proof. Let $x(t)$ be a solution of the problem 1.1. Then from Lemma 1.4, we obtain

$$D^\beta[D^\alpha + \lambda x(t) - I^\beta f(.,x(.)x(t))] = 0.$$

Now by applying Lemma 1.5, we deduce

$$(D^\alpha + \lambda)x(t) - I^\beta f(.,x(.))(t) = c_0 + c_1 t,$$

or equivalently,

$$D^\alpha\Big(x(t) + \lambda I^\alpha x(.)(t) - I^{\alpha+\beta} f(.,x(.))(t) - c_0\frac{t^\alpha}{\Gamma(\alpha+1)} - c_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\Big) = 0.$$

Applying Lemma 1.5 again, the general form of the problem (1.1) can be written as

$$x(t) = I^{\alpha+\beta} f(.,x(.))(t) - \lambda I^\alpha x(.)(t) + c_3 + c_0\frac{t^\alpha}{\Gamma(\alpha+1)} + c_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{2.2}$$

By using the boundary conditions for the problem (1.1), we have

$$\begin{aligned} c_0 &= \Gamma(\alpha+1)\int_0^\eta x(\tau)d\tau - \Gamma(\alpha+1) + \frac{\Gamma(\alpha+1)}{\Gamma(\beta-\alpha)}\Big[\frac{\Gamma(2-\alpha)}{\Gamma(\beta-\alpha)} \\ &. \int_0^1 (1-\tau)^{\beta-\alpha-1} f(\tau,x(\tau))d\tau + \Gamma(1-\alpha)\Big] + \frac{\lambda\Gamma(\alpha+1)}{\Gamma(\alpha)}\int_0^\eta (1-\tau)^{\alpha-1}x(\tau)d\tau \\ &- \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)}\int_0^1 (1-\tau)^{\alpha+\beta-1} f(\tau,x(\tau))d\tau, \\ c_1 &= -\frac{\Gamma(2-\alpha)}{\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1} f(\tau,x(\tau))d\tau, \\ c_2 &= 0, \qquad c_3 = 1. \end{aligned} \tag{2.3}$$

Substituting the values of c_0, c_1, c_2 and c_3 in (2.2), we obtain the solution (2.1). On the other hand, it is easy to prove that, if $x(t)$ is a solution of the integral equation (2.1), then $x(t)$ is also a solution of the problem (1.1). $\square$

3. Investigation in the cases $\lambda \geq 0, \lambda < 0$

To prove the main results, we need the following assumptions:
(H_1) $f : [0,1]\times \to$ be a function such that $f(.,x(.)) \in C([0,1],)$ for each $x \in C([0,1],)$.
(H_2) There exists $L > 0$ such that $0 \leq f(t,x) - f(t,y) \leq L.(x-y)$ for all $x, y \in$ with $x \geq y$.

Theorem 3.1. *With the assumptions $(H1) - (H2)$, if the the problem (1.1) has a coupled lower and upper solution and $\lambda \geq 0, \Lambda = \max\{\Lambda_1, \Lambda_2\} < 1$, where*

$$\begin{aligned} \Lambda_1 &= \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{2|\lambda|}{\Gamma(\alpha+1)} + \frac{1+\Gamma(2-\alpha)}{\Gamma(\alpha+2)}.\frac{L}{\Gamma(\beta-\alpha-1)}, \\ \Lambda_2 &= \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)}.\frac{L}{\Gamma(\beta-\alpha-1)}. \end{aligned} \tag{3.1}$$

then it has a unique solution in C[0,1].

Theorem 3.2. *With the assumptions $(H1) - (H2)$, if the the problem (1.1) has a coupled lower and upper solution and $\lambda < 0, \Lambda < 1$, where*

$$\begin{aligned} \Lambda_1 &= \frac{2.L}{\Gamma(\alpha+\beta+1)} + \frac{2|\lambda|}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)}\frac{2L}{\Gamma(\beta-\alpha+1)}, \\ \Lambda_2 &= \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)}\frac{2L}{\Gamma(\beta-\alpha+1)}. \end{aligned}$$

then it has a unique solution in $C([0,1],\mathbb{R})$.

References

1. R. P. Agarwal, M. Benchohra, S .Hamani, *A survey on existence resultes for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010) 973–1033.
2. A. Amini-Harandi, H. Emami, *A fixed point theorem for contractiontype maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal. **72** (2010) 2238–2242.
3. H. Baghani, *Existence and uniqueness of solutions to fractional Langevin equations involving tow fractional orders. J. Fixed point Theory Appl*, **20** (2018) 4256–4262.
4. O. Baghani, *On fractional Langevin equation involving two fractional orders*, Commun. Nonlinear Sci. Numer. Simulat, **42** (2017) 675–681.